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THE THERMODYNAMICS OF THERMAL
INSTABILITY IN LIQUIDS

By HAROLD JEFFREYS

(*St. John's College, Cambridge*)

SUMMARY

The principle of a stationary eigenvalue found by Jeffreys and Bland in the problem of a fluid sphere heated within is shown to be an expression of the principle that in marginal instability the supply of energy by expansion must just balance the dissipation. The principle is most naturally expressed in terms of the disturbance of temperature; it is equivalent to the forms given by Pellew and Southwell and by Chandrasekhar, but these obscure its meaning, since the use of the radial velocity as the independent variable leads to an interchange of gravity and viscosity as factors on the two sides of the equation.

1. PELLEW AND SOUTHWELL (1) showed that for a layer of liquid heated below the condition for marginal instability could be expressed by a principle of the general type of Rayleigh's principle. Miss Bland (now Mrs. Edwards) and I found a similar result for a liquid sphere heated within (2). I extended this to a sphere with a dense nucleus (3). In both cases the expression on one side of the equation contained the viscosity as a factor, while that on the other contained gravity. This suggests that the principle of a stationary eigenvalue may be a way of expressing the principle that for marginal instability the new energy supplied by volume changes must balance the dissipation by viscosity. A direct proof that it does express such a principle turns out to be rather difficult, but seemed worth attempting. Types of instability other than cellular convection are now known to exist (4), and a completely general physical principle might be useful in suggesting how these types arise.

2. The following discussion refers to the problem discussed by Bland and the author. It is essentially a test of freedom from contradiction; the motion is taken to be a steady departure from symmetry, satisfying the condition of marginal instability. If the conditions are consistent, the thermodynamic supply of energy should balance the dissipation.

Our equation was, for a conducting free surface,

$$ n(n+1)\frac{\lambda}{h^6} \int\int\int \left(\frac{\partial V'}{\partial x_i}\right)^2 d\tau = \int\int\int (\nabla^4 V')^2\, d\tau - \frac{2}{h} \int\int \left(\frac{\partial \nabla^2 V'}{\partial r}\right)^2 dS, \quad \text{(A)} $$

where V' is the disturbance of temperature, h the radius, and

$$ \frac{\lambda}{h^6} = \frac{2\alpha\beta}{\kappa\nu} \frac{g(h)}{h}. \quad \text{(B)} $$

[Quart. Journ. Mech. and Applied Math., Vol. IX, Pt. 1 (1956)]

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2.1. When the outer surface is rigid the surface integral does not arise. We have the relations (subject to certain approximations)

$$x_i u_i = q = -\kappa \nabla^2 V'/2\beta = q_n K_n = q_n r^n S_n, \tag{1}$$

$$\rho' = -\alpha \rho_c V', \tag{2}$$

$$\partial u_i/\partial x_i = \Delta = \alpha \kappa \nabla^2 V', \tag{3}$$

$$u_i = F \frac{\partial K_n}{\partial x_i} + G x_i K_n, \tag{4}$$

$$n(n+1)F = r \frac{dq_n}{dr} + (n+1)q_n, \tag{5}$$

$$(n+1)Gr^2 = -r \frac{dq_n}{dr}. \tag{6}$$

The rate of dissipation is

$$\Phi = 2\rho\nu \iiint (e'_{ik})^2 \, d\tau = \rho\nu \iint S_n^2 \, d\omega \int_0^h r^{2n+2}\{n(n-1)(n+1)(n+2)G_1^2 + \\ + n(n+1)G_2^2 + \tfrac{4}{3}G_3^2\} \, dr, \tag{7}$$

with

$$G_1 = \frac{F}{r^2} = \frac{1}{n(n+1)}\left(\frac{q'_n}{r} + \frac{(n+1)q_n}{r^2}\right) \tag{8}$$

$$G_2 = \frac{F'}{r} + G + 2(n-1)\frac{F}{r^2} = \frac{1}{n(n+1)}\left(q''_n + \frac{2nq'_n}{r} + \frac{2(n^2-1)q_n}{r^2}\right) \tag{9}$$

$$G_3 = rG' + n\left(\frac{F'}{r} + G\right) + \frac{3}{2}\frac{n(n-1)}{r^2}F = \frac{3}{2}\left(\frac{q'_n}{r} + (n-1)\frac{q_n}{r^2}\right). \tag{10}$$

Now

$$\nabla^2 q = \left(q''_n + 2(n+1)\frac{q'_n}{r}\right)r^n S_n, \tag{11}$$

whence

$$\Phi = \frac{\rho\nu}{n(n+1)} \iiint (\nabla^2 q)^2 \, d\tau + \frac{\rho\nu}{n(n+1)} \iint S_n^2 \, d\omega \int_0^h r^{2n+2}\{\} \, dr,$$

where

$$\{\} = 4\left(\frac{-q'_n}{r} + \frac{(n^2-1)q_n}{r^2}\right)\left(q''_n + (2n+1)\frac{q'_n}{r} + (n^2-1)\frac{q_n}{r^2}\right) + \\ + (n-1)(n+2)\left(\frac{q'_n}{r} + (n+1)\frac{q_n}{r^2}\right)^2 + 3n(n+1)\left(\frac{q'_n}{r} + \frac{(n-1)q_n}{r^2}\right)^2. \tag{12}$$

The integral with respect to r is

$$\int_0^h r^{2n+1}\left(-4q''_n q'_n + 4(n^2-1)\frac{q''_n q_n}{r} + (4n^2-4n-6)\frac{q'^2_n}{r} + \right. \\ \left. + 4(n^2-1)(4n+1)\frac{q_n q'_n}{r^2} + 2(n^2-1)(4n^2-1)\frac{q_n^2}{r^3}\right) dr \tag{13}$$

$$= \left[-2r^{2n+1}q'^2_n + 4(n^2-1)r^{2n}q'_n q_n + (n^2-1)(4n+2)r^{2n-1}q_n^2\right]_{r=h} \tag{14}$$

after two integrations by parts. This vanishes at a rigid surface, since q and $\partial q/\partial r$ vanish. At a free surface $q = 0$, and

$$\partial q/\partial r = q'_n r^n S_n. \tag{15}$$

Hence for a rigid boundary the dissipation is

$$\Phi = \frac{\rho\nu}{n(n+1)} \iiint (\nabla^2 q)^2 \, d\tau \tag{16}$$

and for a free surface it is

$$\Phi = \frac{\rho\nu}{n(n+1)} \left\{ \iiint (\nabla^2 q)^2 \, d\tau - \frac{2}{h} \iint \left(\frac{\partial q}{\partial r}\right)^2 dS \right\}. \tag{17}$$

It follows from the form of q that in each case the right side of (A) is

$$\frac{n(n+1)}{\rho\nu} \left(\frac{2\beta}{\kappa}\right)^2 \Phi. \tag{18}$$

Incidentally we failed previously to prove that the right side of (A) is positive definite and consequently it might have been possible for a wildly wrong form of V' to yield a negative β. This is now seen to be impossible, since the dissipation is a sum of integrals of squares.

For the rate of generation of energy by expansion with tension we have

$$\Psi = \iiint \tfrac{1}{3} p_{mm} \frac{\partial u_i}{\partial x_i} \, d\tau = \iint \tfrac{1}{3} p_{mm} l_i u_i \, dS - \iiint u_i \frac{\partial}{\partial x_i} (\tfrac{1}{3} p_{mm}) \, d\tau \tag{19}$$

and the integrated part vanishes since $q = 0$ at the boundary. By the equations of motion, apart from a small term in the viscosity,

$$\tfrac{1}{3} p_{mm} = -p, \tag{20}$$

$$\frac{\partial p}{\partial x_i} = \rho_c \frac{\partial U'}{\partial x_i} - g \frac{x_i}{r} \rho'. \tag{21}$$

Since
$$\rho' = -\rho_c \alpha V, \quad \text{and} \quad g/r \doteq g(h)/h, \tag{22}$$

$$\Psi = \iiint u_i \left(\rho_c \frac{\partial U'}{\partial x_i} + \frac{g(h)}{h} x_i \rho_c \alpha V' \right) d\tau. \tag{23}$$

The first part is the rate of increase of the part of the work function due to the deformation, and vanishes if the motion is steady. The second part is

$$\Psi = \iiint \frac{g(h)}{h} \rho_c \alpha q V' \, d\tau$$

$$= - \iiint \frac{g(h)}{h} \rho_c \alpha \frac{\kappa}{2\beta} V' \nabla^2 V' \, d\tau$$

$$= \frac{g(h)}{h} \rho_c \frac{\alpha\kappa}{2\beta} \iiint \left(\frac{\partial V'}{\partial x_i}\right)^2 d\tau \tag{24}$$

since $V' = 0$ at the surface. Thus the left of (A) is $n(n+1)(2\beta/\kappa)^2/\rho_c$ times the rate of generation of energy. Hence the conjecture that (A) is equivalent to the thermodynamic principle is correct.

It was hoped that the thermodynamic argument would yield a more general principle that could be used to test stability under other types of disturbance than those contemplated here, and possibly under more complicated conditions (for instance, when the variation of density in equilibrium is not small). The argument is capable of extension, but in the present problem it leads to no important simplification, since all the approximations used in deriving (A) have been used again here. Previous methods, however, depend on rather severe simplifications, and it is possible that in problems where these are impossible direct use of the thermodynamic principle may succeed.

3. The problem of cellular convection in a sphere has been rediscussed by S. Chandrasekhar (5), who uses a principle of a stationary eigenvalue that is superficially different from ours. But if in (A) we use the differential equation

$$\nabla^6 V' = -n(n+1)\frac{\lambda}{h^6} V' \tag{25}$$

we get

$$\iiint \left(\frac{\partial}{\partial x_i}(\nabla^6 V')\right)^2 d\tau = n(n+1)\frac{\lambda}{h^6}\left\{\iiint (\nabla^4 V')^2 d\tau - \frac{2}{h}\iint \left(\frac{\partial \nabla^2 V'}{\partial r}\right)^2 dS\right\}. \tag{26}$$

Since q is proportional to $\nabla^2 V'$, this can be seen immediately to be the same as Chandrasekhar's equation, which is expressed in terms of q. But since λ has been transferred to the opposite side of the equation, the counterparts of the rates of generation and dissipation of energy now contain ν and g respectively instead of g and ν. The physical meaning of the equation is thus obscured. This is a consequence of the use of q instead of V' as the fundamental dependent variable. Rayleigh used the vertical velocity as his fundamental variable, and has been followed by several other writers. Kelvin, however, pointed out that in problems of elasticity the temperature enters in much the same way as the gravitation potential, and there are problems where these are given. If the temperature is the last variable to be retained the technique used for the theory of the straining of an elastic sphere can be used with little change.

We proceeded (for a complete sphere) by assuming a cubic in r^2 for V_n, where $V' = V_n K_n$. In this form (A) succeeds, but $\nabla^6 V'$ would not vanish at the boundary as it should, and if (26) is used a different form is needed. Chandrasekhar takes $\nabla^6 V'$ to be a multiple of $J_{n+\frac{1}{2}}(\alpha r)S_n r^{-\frac{1}{2}}$, α being chosen so that $\nabla^6 V'$ vanishes at the outer surface. He then finds $\nabla^2 V'$ by solving

the differential equation with the boundary conditions on $\nabla^2 V'$. The results are more accurate than ours, as he shows by taking a second approximation including a second term containing another root of the equation $J_{n+\frac{1}{2}}(\alpha h) = 0$. But since the left side of his equation depends wholly on the trial solution this can only be because, in fact, he has taken a very accurate trial solution; and had he carried the integration one stage further so as to get a form for V', and then used (A), his results would presumably have been more accurate still.

Chandrasekhar remarks that our way of getting the conditions at the surface is 'somewhat less direct' than his, which uses spherical coordinates. It is well known, however, that the equations in spherical polar coordinates are less convenient than those in rectangular ones in this type of problem. It seems pointless to use the polar equations for the boundary conditions when they have been avoided in getting the equations for the interior. The functions F, G of (4), introduced in Love's *Geodynamics*, give the boundary conditions easily without this transformation.

4. The adaptation of (A) to the case of a horizontal layer of fluid is straightforward, since in the problem of a sphere the inner boundary may be arbitrarily near the outer one. In the spherical problem

$$\lambda = -\frac{\alpha g(h)}{\kappa \nu} h^4 \left(\frac{\partial V_0}{\partial r} \right)_{r=h}$$

and for the plane one

$$\lambda_P = -\frac{\alpha g h_P^4}{\kappa \nu} \frac{\partial V_0}{\partial r}.$$

Thus taking the inner boundary as $r = h - h_P$ and treating h_P/h as small we find that the surface integral tends to zero and, if the disturbance of temperature is $Z \cos lx \cos my$, with

$$\zeta = Z/h_P, \qquad (l^2 + m^2) h_P = a^2,$$

$$a^2 \lambda_P \int \left\{ \left(\frac{dZ}{d\zeta} \right)^2 + a^2 Z^2 \right\} d\zeta = \int \left\{ \left(\frac{d^2}{d\zeta^2} - a^2 \right)^2 Z \right\}^2 d\zeta.$$

This is easily seen to be equivalent to the stationary eigenvalue condition given by Pellew and Southwell.

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THE VARIATIONAL METHOD IN HYDRODYNAMICS

By P. E. LUSH and T. M. CHERRY

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SUMMARY

The problem to be considered is to determine the steady irrotational isentropic motion of a compressible fluid through a region when the normal mass-flow (i.e density times normal velocity) is prescribed over the boundary. For this hydrodynamic problem there are given two equivalent problems of the calculus of variations; the integral to be made stationary depends in one case on the velocity potential, and in the other case on the stream function. In the first instance it is required that the region be bounded; but it is shown how the variational formulation may be modified so as to apply to an unbounded region. Here, for definiteness, the case is taken of streaming motion past an aerofoil. Finally it is shown how approximate solutions of the variational problem can be obtained by direct numerical (Rayleigh–Ritz) procedure. Numerical results are given for streaming past a circular cylinder and are compared with those found for the same problem by other methods.

1. Introduction

THE practical solution of a problem, concerning the flow of an inviscid compressible fluid in a region R, by direct variational methods can be divided into four stages:

(i) To find integrals whose variation leads to field equations (Euler equations) equivalent to those of hydrodynamics.

(ii) To add to such an integral, if necessary, supplementary terms so as to fit the boundary conditions of the problem. This will give a variand which suits an arbitrary region R, provided the region is bounded; but if a boundary of R moves off to infinity the variand may diverge, and so we require:

(iii) To adjust the variand so that in the limiting case it remains finite and suits the prescribed conditions of the problem at infinity, while still giving the correct field equations and conditions at ordinary boundary points.

(iv) To find, by algebraic and numerical calculation, field-functions which approximately minimize or make stationary the variand; the ideas and technique here are analogous to those of the Rayleigh–Ritz method as used, say, for problems in elasticity, but the hydrodynamic application is more complicated.

One might add an intermediate stage: to prove that the variational problem, as formulated after stage (ii) or (iii), has a well-determined

solution. It is likely that this stage is very difficult, and it remains virtually unexplored. We shall be content with the ordinary sort of judgement as to 'proper formulation' of a problem; it is, of course, necessary that a problem should pass this test before one would attack it at stage (iv).

Regarding stage (i) the integrals we shall need were given by Bateman (1). On the other stages, however, the literature seems to contain little that is in definitive shape. The most relevant paper seems to be that of Wang (2), but at stage (iii) he does not properly handle the convergence problem, and at stage (iv) his work applies only to gases for which the pressure p and density ρ are connected by $p\rho^{-\gamma} = $ constant with $\gamma/(\gamma-1)$ *integral*.

For stage (ii) we give two variational formulations of the problem of steady irrotational isentropic flow of a compressible fluid through a bounded region when the rate of normal mass-flow across its boundary is prescribed; the case of a rigid boundary-arc is hereby covered. If the flow is plane and everywhere subsonic, the first formulation is to maximize an integral J_1 and the second is to minimize an integral J_2, and the stationary values of J_1 and J_2 are equal. Hence, from the difference $J_2 - J_1$, we have a common-sense criterion for comparing the overall accuracy of two or more approximate solutions. For stages (iii) and (iv) we consider for definiteness (following Wang) the flow of a polytropic gas ($p\rho^{-\gamma} = $ constant) past a cylindrical obstacle. We give a strict treatment of the convergence problem that arises because of the infinite extent of the flow-field, and show how to handle the numerical approximations for the case where γ is arbitrary.

Numerical results are given for the flow of an infinite stream, without circulation, past a circular cylinder, the chosen data being $\gamma = 1\cdot405$ and Mach number at infinity $= 0\cdot4$. Evidence is presented that the results are correct to about 1 part in 200. The whole flow, as calculated, is subsonic, but M rises to $0\cdot99$ or $0\cdot995$ at the ends of the transverse axis. The results agree, to about 1 per cent., with those found by Imai (3) using the Rayleigh–Janzen method, with the same data; and there is a satisfactory check also with the results found by Taylor and Sharman (4) by the electrolytic tank method.

The variational procedure can be applied also, with apparent success, to supersonic and transonic problems. We have constructed the theory for what we call stage (iii) for the case of a nozzle bounded by a pair of hyperbolas, and have applied it numerically to a case in which there are limited supersonic regions at the sides of the throat.

2. Steady motion with prescribed normal mass-flow at the boundary

For the steady irrotational isentropic motion of a compressible inviscid

fluid, Bateman has given two integrals whose variation leads to the hydro-dynamic field equations. We shall show how to amplify his investigation to suit the boundary condition of prescribed normal mass-flow. For intelligibility we give an *ab initio* treatment, which in any case is quite short.[†]

Using standard notation, let $U(\rho)$ be the internal energy per unit mass so that

$$U(\rho) = \int^{\rho} \frac{p \, d\rho}{\rho^2}, \tag{1}$$

where p is a function of ρ only. Then for the field equations we can take Bernoulli's integral

$$\tfrac{1}{2} u_r u_r + (\rho U)' = 0, \tag{2}$$

and the continuity equation

$$\frac{\partial(\rho u_r)}{\partial x_r} = 0; \tag{3}$$

r runs through the values $1, 2, 3$, the summation convention is used, and the accent $(\ )'$ denotes the ρ-derivative $d(\ )/d\rho$.

Use of the velocity potential. The function-field consists of all velocity-potentials of class C_3,[‡] and corresponding to any such ϕ we *define* u_r by

$$u_r = \partial\phi/\partial x_r = \phi_{x_r}, \tag{4}$$

and then ρ by equation (2) and p by (1). The continuity equation (3) will then usually not be satisfied, and our object is to give an integral whose variation yields (3) as its field equation. Bateman (1) showed that we can take

$$J[\phi] = \int_R p \, d\tau, \tag{5}$$

where the region, R, of integration has its boundary surface B sufficiently regular to permit the application of Green's theorem.

From (1) we have $p = \rho^2 U'$, and thence

$$p' = 2\rho U' + \rho^2 U'' = \rho(\rho U)'' = c^2, \text{ say}, \tag{6}$$

and from (2), writing x for any of the x_r,

$$\frac{\partial p}{\partial \phi_x} = \rho \frac{\partial(\rho U)'}{\partial \phi_x} = -\rho\phi_x.$$

Thence,
$$\frac{\partial \rho}{\partial \phi_x} = -\frac{\rho\phi_x}{c^2},$$

---

† Bateman's treatment is in places obscure: he does not always make it clear whether his equations are asserted for all the functions of the field or only for the extremal functions.

‡ The precise restriction as to 'regularity' of the field is here of little importance; we have chosen class C_3 so as to be able to exhibit Taylor expansions correct to the second order.

and

$$\frac{\partial^2 p}{\partial \phi_x^2} = -\rho - \phi_x \frac{\partial \rho}{\partial \phi_x} = -\frac{\rho}{c^2}(c^2 - \phi_x^2),$$

$$\frac{\partial^2 p}{\partial \phi_{x_1} \partial \phi_{x_2}} = \frac{\rho \phi_{x_1} \phi_{x_2}}{c^2}.$$

Hence, if $\phi + \lambda \Phi$ is any family of functions of the field, where we regard ϕ, Φ as fixed and λ as a small parameter, Taylor's theorem gives

$$\delta p = p[\phi + \lambda \Phi] - p[\phi] = -\lambda \rho u_r \Phi_{x_r} - \tfrac{1}{2}\lambda^2 \rho Q/c^2 + O(\lambda^3), \qquad (7\,\text{a})$$

where

$$Q = c^2 \Phi_{x_r} \Phi_{x_r} - u_r u_s \Phi_{x_r} \Phi_{x_s}, \qquad (7\,\text{b})$$

and ρ, u_r, c^2 are the functionals of ϕ defined above.

From (5) we have $\delta J = \int \delta p \, d\tau$. Substituting for δp and applying Green's theorem to the term in λ we get

$$\delta J = J[\phi + \lambda \Phi] - J[\phi]$$
$$= -\lambda \int_B \rho \Phi l_r u_r \, dS + \lambda \int_R \Phi \frac{\partial(\rho u_r)}{\partial x_r} \, d\tau - \tfrac{1}{2}\lambda^2 \int_R \frac{\rho Q}{c^2} \, d\tau + O(\lambda^3),$$

where l_r gives the direction-cosines of the outward normal to B. The surface integral vanishes if $\Phi = 0$ or $l_r u_r = 0$; but $\Phi = 0$ corresponds to the assignment of ϕ on B, which is not physically acceptable as a boundary condition. To eliminate this term let

$$J_1[\phi] = J[\phi] + \int_B \rho \phi l_r u_r \, dS. \qquad (8)$$

Then, writing $\bar{\phi}$ for $\phi + \lambda \Phi$ and $\bar{\rho}$, $\bar{u}_r$ for the corresponding values of ρ, u_r,

$$\delta J_1 = \int_B \bar{\phi}(\bar{\rho}\bar{u}_r - \rho u_r)l_r \, dS + \lambda \int_R \Phi \frac{\partial(\rho u_r)}{\partial x_r} \, d\tau - \tfrac{1}{2}\lambda^2 \int_R \frac{\rho Q}{c^2} \, d\tau + O(\lambda^3).$$

We now restrict the field to *all those functions ϕ of class C_3 for which $\rho u_r l_r$ takes a prescribed value at each point of B*, and deduce:†

A field-function ϕ (if any) which makes $J_1[\phi]$ stationary specifies a flow for which the continuity equation (3) is satisfied throughout R; and the stationary value will be a maximum if the quadratic form Q in Φ_{x_r}, defined by (7 b), is positive-definite, which is so if $u_r u_r < c^2$, i.e. if the stationary flow is subsonic throughout R.

The quantity $\rho u_r l_r$ prescribed on the boundary is the 'normal mass-flow', or more fully, the time-rate of normal mass-flow per unit area; on a rigid boundary-arc this is zero.

Of course, we have not proved that $J_1[\phi]$ does take a stationary value within the restricted field $\phi(x_r)$, i.e. that the field includes an extremal function. A necessary condition for this is that the prescribed boundary

† From the fundamental lemma of the Calculus of Variations.

values of $\rho u_r l_r$ be consistent with the validity throughout R of the continuity equation, which will be so *if the total mass-flow across the boundary, as prescribed, is zero*. If the fluid were incompressible this condition would also be sufficient for the existence of an extremal. But for a compressible fluid it seems to be still unknown whether this condition is by itself sufficient: there may be required a further condition whose effect is to restrict the extremal flow to be everywhere subsonic. In addition there is the question of uniqueness of the proposed extremal; for this we may guess (from analogy with the incompressible case) that it is sufficient that R be simply-connected; some remarks concerning multiply-connected regions are given later.

Since the data and field equations involve ϕ only through its derivatives, it may appear paradoxical that the surface integral in (8) should involve the factor ϕ. But the only alteration in a function ϕ which leaves its derivatives unaffected is the addition of a constant C, and the consequential alteration in the surface integral is

$$C \int_B \rho l_r u_r \, dS = C \int_B (\rho l_r u_r)_{\text{prescribed}} \, dS,$$

which is zero because of the consistency condition just introduced.

3. Use of the stream function

Confining attention now to steady plane flows we use the notation x, y, u, v, q in place of the suffix notation. Bateman's second integral is

$$J_2[\psi] = \iint_R (p + \rho q^2) \, dx dy, \tag{9}$$

where R is a bounded region, supposed at present simply-connected, whose bounding curve C is sufficiently smooth for the application of Green's theorem. The field consists in the first instance of all functions $\psi(x, y)$ of class C_3, and for any such ψ we define ρ, u, v, q, p by

$$\left. \begin{array}{ccc} \rho u = \dfrac{\partial \psi}{\partial y}, & \rho v = -\dfrac{\partial \psi}{\partial x}, & q = +\sqrt{(u^2 + v^2)}, \qquad p = \rho^2 U', \\[2mm] & \tfrac{1}{2} q^2 + (\rho U)' = 0. \end{array} \right\} \tag{10}$$

The equation for ρ is thus

$$\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + 2\rho^2 (\rho U)' = 0,$$

where the derivative of $\rho^2(\rho U)'$ vanishes when $\rho(\rho U)'' + 2(\rho U)' = 0$, and so by (6) and (10) when $q^2 = c^2$; hence $\rho[\psi]$ is singular when $q = c$, and this 'sonic singularity' is latent throughout all that follows.

To any ψ corresponds a fluid motion (u, v, ρ) satisfying the continuity equation, but the p defined by $(10)_4$† will not in general 'fit' this motion. For any such motion, $(10)_{1,2}$ give $\mathbf{t} \cdot \nabla\psi = \rho\mathbf{n} \cdot \mathbf{q}$, where $\mathbf{t}$, $\mathbf{n}$ denote unit vectors tangential and normal to C; and on the right we have the normal mass-flow across C, so that when this is prescribed $\partial\psi/\partial s$ is prescribed, and ψ is determined on C to an arbitrary constant. Moreover, ψ will be single-valued on C provided that $\int_C \rho\mathbf{n} \cdot \mathbf{q} \, ds = 0$, and since R is simply-connected this is the consistency condition introduced in section 2. Hence the boundary-value problem where the normal mass-flow is prescribed, already treated in section 2, will be suited *if we restrict the field to those (single-valued) functions ψ that take prescribed values on the boundary C of R.*

From $(10)_4$ we have $dp = \rho d(\rho U)'$, and thence from $(10)_5$

$$d(p + \rho q^2) = -\rho q \, dq + \rho q \, dq + q d(\rho q) = u d(\rho u) + v d(\rho v),$$

so by $(10)_{1,2}$

$$\frac{\partial(p + \rho q^2)}{\partial\psi_x} = -v = \frac{\psi_x}{\rho}, \qquad \frac{\partial(p + \rho q^2)}{\partial\psi_y} = u = \frac{\psi_y}{\rho}.$$

Also, from $(10)_5$,

$$\rho(\rho U)'' d\rho = -\rho q \, dq = -\rho(u \, du + v \, dv)$$
$$= -u(d\psi_y - u \, d\rho) + v(d\psi_x + v \, d\rho),$$

so by (6)

$$\frac{\partial\rho}{\partial\psi_x} = \frac{v}{c^2 - q^2} = -\frac{\psi_x}{\rho(c^2 - q^2)}, \qquad \frac{d\rho}{d\psi_y} = -\frac{u}{c^2 - q^2} = -\frac{\psi_y}{\rho(c^2 - q^2)}.$$

Thence follow

$$\frac{\partial^2(p + \rho q^2)}{\partial\psi_x^2} = \frac{c^2 - u^2}{\rho(c^2 - q^2)}, \qquad \frac{\partial^2(p + \rho q^2)}{\partial\psi_y^2} = \frac{c^2 - v^2}{\rho(c^2 - q^2)},$$

$$\frac{\partial^2(p + \rho q^2)}{\partial\psi_x \partial\psi_y} = -\frac{uv}{\rho(c^2 - q^2)}.$$

Hence, if $\psi + \lambda\Psi$ is a family of field-functions in which λ is a small parameter,

$$\delta(p + \rho q^2) = (p + \rho q^2)[\psi + \lambda\Psi] - (p + \rho q^2)[\psi]$$
$$= \lambda(u\Psi_y - v\Psi_x) + \frac{\lambda^2 Q}{2\rho(c^2 - q^2)} + O(\lambda^3), \tag{11}$$

where $\qquad Q = (c^2 - u^2)\Psi_x^2 - 2uv\Psi_x\Psi_y + (c^2 - v^2)\Psi_y^2. \tag{12}$

On carrying the development (11) into (9) and applying Green's theorem to the coefficient of λ we obtain

$$\delta J_2 = J_2[\psi + \lambda\Psi] - J_2[\psi]$$
$$= \lambda \int_C \mathbf{t} \cdot \mathbf{q}\Psi \, ds + \lambda \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\Psi \, dx dy + \tfrac{1}{2}\lambda^2 \iint_R \frac{Q \, dx dy}{\rho(c^2 - q^2)} + O(\lambda^3). \tag{13}$$

† $(10)_4$ denotes the fourth equation in (10).

By the boundary conditions imposed on the field, Ψ is zero on C; so *if any ψ makes $J_2[\psi]$ stationary it gives a flow for which*

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0, \tag{13a}$$

throughout R, i.e. an irrotational flow. From (10) and (13a) such a flow satisfies Euler's dynamical equations, and from (12), (13) if this flow is everywhere subsonic it minimizes J_2.

Return now to equation (8) and, taking the plane case, let ϕ be the extremal function ϕ_{extr} (assumed to exist). Since this function satisfies the continuity equation the boundary integral is equal to

$$\iint_R (\rho q^2)_{\text{extr}} \, dxdy,$$

so the extremal ϕ and the extremal ψ give

$$J_1[\phi_{\text{extr}}] = J_2[\psi_{\text{extr}}] = \iint_R (p + \rho q^2) \, dxdy. \tag{14}$$

In the subsonic case, however, J_1 is a maximum and J_2 a minimum, so for the problem of irrotational motion with prescribed normal mass-flow we have two distinct formulations $\delta J_1 = 0$, $\delta J_2 = 0$.† It follows that if ϕ, ψ give 'approximate solutions' of this boundary-value problem, obtained, say, by numerical processes, then

$$J_1[\phi] < J_1[\phi_{\text{extr}}] < J_2[\psi].$$

For comparing the relative accuracy of two or more approximate solutions ϕ we can take $J_1[\phi_{\text{extr}}] - J_1[\phi]$ as, in a vague sense, a 'criterion of mean error'; and similarly for ψ. At a similar level of vagueness we can estimate whether an approximation ϕ and an approximation ψ are of 'comparable accuracy', e.g. they might be obtained by Rayleigh–Ritz procedures using the same number of adjustable constants. In such a case we could equally well take $J_2[\psi] - J_1[\phi]$ as the error criterion; and this, unlike the criterion involving $J_1[\phi_{\text{extr}}]$, is obtainable from knowledge only of the approximations ϕ, ψ.

Multiply-connected regions. Let E be the external boundary and $C_1, C_2, \ldots$ the internal ones. Here usually it will be necessary to use multiple-valued functions ϕ, ψ, and from what is known for an incompressible fluid we shall have to assign the circulations κ_r round the C_r (in addition to the normal

† The sonic singularity which, as has been seen, attaches to the formulation $\delta J_2 = 0$ is suggestive in relation to the notorious problem of the existence of transonic flows through regions with assigned boundaries.

mass-flow) to get a determinate problem. Let μ_r be the total normal mass-flow across C_r and (x_r, y_r) a point within C_r. Then the field-functions ϕ can be restricted to the form

$$\phi = \sum_r \frac{\kappa_r}{2\pi} \arctan \frac{y - y_r}{x - x_r} + \bar{\phi}, \tag{15}$$

where $\bar{\phi}$ is single-valued. From (8) J_1 is multiple-valued, but the difference between any two of its determinations is

$$\sum_r n_r \kappa_r \int_{C_r} \rho l_r u_r \, ds \quad (n_r \text{ integral}).$$

This is $\sum n_r \kappa_r \mu_r$, and since μ_r is a datum, δJ_1 is single-valued; and it is only δJ_1, not J_1, which is significant.

For ψ we must take a form like (15) with μ_r in place of κ_r, and there is the further feature that the boundary values of ψ enter into the definition of the field. These boundary values are obtained by integrating the datum $\partial \psi / \partial s$ round E and each C_r, so on each of these curves the data determine ψ only to an additive const. nt, which remains so far unknown. The constant attached to E may be dropped, but the field must include 'all' functions for which, on each C_r, ψ has the form $\psi_r + A_r$, where ψ_r is given and the constant A_r remains arbitrary.

Extensions

The work of this section can be extended in two directions.

(i) *Irrotational flow in three dimensions.* If we replace $(10)_{1,2}$ by

$$\rho \mathbf{q} = \operatorname{curl} \boldsymbol{\psi}$$

and take the three components of the vector function $\boldsymbol{\psi}$ as independently subject to variation, the field-equations obtained from $\delta J_2 = 0$ are easily found to be $\operatorname{curl} \mathbf{q} = 0$, and from this along with (10) follow Euler's dynamical equations.

(ii) Returning to two dimensions, let $(10)_5$ be replaced by

$$\tfrac{1}{2} q^2 + (\rho U)' = f(\psi),$$

where f is a 'given' function, not subject to variation. Then $\delta J_2 = 0$ gives field equations which, when combined with the definitions of ρ, u, v, q, p, imply the satisfaction of Euler's dynamical equations. By proper choice of f we may expect thus to cover *any* case of steady plane rotational motion.

4. Flow in an unbounded region

For definiteness we take the case of plane flow in which a uniform subsonic stream of speed V is locally deflected, without circulation, by the immersion of a bounded obstacle: the aerofoil problem.† The ideas in

† The treatment that follows may be regarded as an underpinning of the investigation of Wang (2). He made the substitution (16) only at the numerical stage of his work, and was content with heuristic evidence for the correctness of his procedure.

what follows can be applied in other cases, e.g. to flow in an infinite nozzle, but the details for each case are distinct.

For a region R bounded internally by a curve C_0 and externally by a circle C_R of large radius R, the procedure of section 2 involves the maximization of J_1, where

$$J_1[\phi] = \iint_R p\,dxdy + \int_{C_R, C_0} \rho\phi\frac{\partial\phi}{\partial n}\,ds;$$

and by (14), $J_1[\phi_{\mathrm{extr}}] = \iint_R (p + \rho q^2)\,dxdy$. Thus, for all functions ϕ which are close to the extremal function, $J_1[\phi]$ will tend to ∞ as $R \to \infty$. To circumvent this difficulty the ideas are (i) that the proposed extremal flow will be given by

$$\phi = Vx + \chi, \tag{16}$$

where χ is small when $r = \sqrt{(x^2 + y^2)}$ is large, and hence that we can restrict the field to functions of this sort; and (ii) that the principal part of $J_1[\phi]$ will be independent of χ, so that it can be subtracted without affecting δJ_1: more generally, we can subtract a part whose variation vanishes at infinity (only). Accordingly† we define

$$J_R[\phi] = \iint_R (p - p_\infty)\,dxdy + \int_{C_R} \chi\rho_\infty\frac{\partial(Vx)}{\partial n}\,ds + \int_{C_0} \phi\rho\frac{\partial\phi}{\partial n}\,ds, \tag{17}$$

and, for a suitably restricted field ϕ, or χ, we shall show (i) that as $R \to \infty$, $J_R[\phi]$ converges to a limit $J_\infty[\phi]$, and (ii) that for $\delta J_\infty[\phi] = 0$ it is necessary that the continuity equation holds throughout the region exterior to C_0.

Regarding a subsonic stream, without circulation, which tends to uniformity at infinity, it is known, or at least may be plausibly assumed, that when r is large

$$\phi = Vx + \frac{f_1(\theta)}{r} + \frac{f_2(\theta)}{r^2} + \ldots = Vx + \chi,$$

where $f_1, f_2, \ldots$ are trigonometric series in the polar angle θ, and that the series converges uniformly when $|\mathrm{im}\,\theta|$ and r^{-1} are sufficiently small.‡ Accordingly we confine the field ϕ to functions having this behaviour when r is large, with the coefficients in the f_n remaining as arbitrary as is consistent with uniform convergence of the series over the whole field. Then, for a linear family $\phi = \phi_0 + \lambda\Phi$ (or $\chi = \chi_0 + \lambda\Phi$), λ will be involved linearly in the said coefficients, so that differentiation with respect to λ

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† The definition (17) is a reasonable guess, which is justified by its success.

‡ See, for example, Batchelor (5); the leading fact is that the field equation for ϕ is elliptic, so that its solutions are analytic.

will not affect the order of magnitude of a term when r is large. Hence we shall have

$$\chi = O\!\left(\frac{1}{r}\right), \qquad \nabla\chi = O\!\left(\frac{1}{r^2}\right)$$
$$\frac{\partial\chi}{\partial\lambda} = O\!\left(\frac{1}{r}\right), \qquad \frac{\partial}{\partial\lambda}\nabla\chi = O\!\left(\frac{1}{r^2}\right) \tag{18}$$

uniformly over the field and over $0 \leqslant \theta \leqslant 2\pi$. Alternatively, the conditions (18) may be taken directly into the definition of the field (16); and there is the further condition that $\partial\phi/\partial n = 0$ on the internal boundary C_0. The presumption is that the field so defined is sufficiently extensive to include the desired extremal.

In what follows we shall assume that the fluid is a 'polytropic' one for which

$$p\rho^{-\gamma} = \text{constant} = p_\infty\,\rho_\infty^{-\gamma}.$$

This enables us to operate with closed formulae for p, ρ in terms of q, viz.

$$p = p_0\!\left(1 - \frac{q^2}{2\beta c_0^2}\right)^{\beta\gamma}, \qquad \rho = \rho_0\!\left(1 - \frac{q^2}{2\beta c_0^2}\right)^{\beta}, \qquad \beta = \frac{1}{\gamma-1}, \tag{19}$$

where the zero suffix denotes stagnation values; when the $p\rho$-relation is left arbitrary the work involves additional algebraic detail and approximation. From (16), (18) we have

$$q^2 = V^2 + 2V\,\partial\chi/\partial x + (\nabla\chi)^2 = V^2 + 2V\,\partial\chi/\partial x + O(r^{-4}),$$

and then from (19)

$$p = p_0\!\left\{1 - \frac{V^2}{2\beta c_0^2} - \frac{V}{\beta c_0^2}\frac{\partial\chi}{\partial x} + O\!\left(\frac{1}{r^4}\right)\right\}^{\beta\gamma}$$
$$= p_\infty\!\left\{1 - \frac{V}{\beta c_\infty^2}\frac{\partial\chi}{\partial x} + O\!\left(\frac{1}{r^4}\right)\right\}^{\beta\gamma}$$
$$= p_\infty\!\left\{1 - \frac{\gamma V}{c_\infty^2}\frac{\partial\chi}{\partial x} + O\!\left(\frac{1}{r^4}\right)\right\}$$
$$= p_\infty - \rho_\infty V\frac{\partial\chi}{\partial x} + O\!\left(\frac{1}{r^4}\right), \tag{20}$$

since $c_\infty^2 = c_0^2 - V^2/2\beta = \gamma p_\infty/\rho_\infty$. Now from (17)

$$J_R[\phi] = \iint\limits_R (p-p_\infty)\,dxdy + \iint\limits_R V\rho_\infty\frac{\partial\chi}{\partial x}\,dxdy + K, \tag{21}$$

where

$$K = \int\limits_{C_0}\left(\rho\phi\frac{\partial\phi}{\partial n} - V\rho_\infty\chi\frac{\partial x}{\partial n}\right)ds. \tag{22}$$

From (20), therefore, $J_R[\phi]$ converges as $R \to \infty$, and

$$J_\infty[\phi] = \iint_\infty \left(p - p_\infty + V\rho_\infty \frac{\partial \chi}{\partial x}\right) dx\,dy + K. \tag{23}$$

Now let ϕ have the form $\phi_0 + \lambda\Phi$, and therefore χ the form $\chi_0 + \lambda\Phi$. Then

$$\frac{\partial}{\partial \lambda}\left(p - p_\infty + V\rho_\infty \frac{\partial \chi}{\partial x}\right) = O\left(\frac{1}{r^4}\right);$$

the proof is similar to that of (20), starting with the closed expression for $\partial p/\partial \lambda$ derived from (19). Hence we can differentiate (23) under the integral sign, and

$$\frac{\partial}{\partial \lambda} J_\infty[\phi] = \lim_{R \to \infty} \iint_R \left(\frac{\partial p}{\partial \lambda} + V\rho_\infty \frac{\partial^2 \chi}{\partial \lambda \partial x}\right) dx\,dy + \frac{\partial K}{\partial \lambda}. \tag{24}$$

But by (7 a),

$$\iint_R \left(\frac{\partial p}{\partial \lambda}\right)_{\lambda=0} dx\,dy = -\int_{C_0, C_R} \rho \Phi \frac{\partial \phi}{\partial n} ds + \iint_R \Phi \operatorname{div}(\rho\nabla\phi)\, dx\,dy,$$

and

$$\iint_R V\rho_\infty \left(\frac{\partial^2 \chi}{\partial \lambda \,\partial x}\right)_{\lambda=0} dx\,dy = \iint_R V\rho_\infty \frac{\partial \Phi}{\partial x} dx\,dy = \int_{C_0, C_R} V\rho_\infty \Phi \frac{\partial x}{\partial n} ds.$$

On C_R, $\rho = \rho_\infty\{1 + O(r^{-2})\}$, and

$$\frac{\partial \phi}{\partial n} = \frac{\partial(Vx)}{\partial n} + \frac{\partial \chi}{\partial n} = \frac{\partial(Vx)}{\partial n} + O(r^{-2}),$$

so

$$\Phi\left\{\rho \frac{\partial \phi}{\partial n} - \rho_\infty \frac{\partial(Vx)}{\partial n}\right\} = O(r^{-3}).$$

Substituting from these formulae into (24), the integrals over C_R combine into one which vanishes in the limit; and, for $\lambda = 0$,

$$\frac{\partial}{\partial \lambda} J_\infty[\phi] = \int_{C_0} \Phi\left(V\rho_\infty \frac{\partial x}{\partial n} - \rho \frac{\partial \phi}{\partial n}\right) ds + \iint_\infty \Phi \operatorname{div}(\rho\nabla\phi)\, dx\,dy + \frac{\partial K}{\partial \lambda};$$

the argument shows that the double integral on the right must converge, and in fact its integrand is easily verified to be $O(r^{-4})$.

In the formula last obtained let the value of $\partial K/\partial \lambda$ be substituted from (22), with $\partial\phi/\partial\lambda = \partial\chi/\partial\lambda = \Phi$. Since $\rho\, \partial\phi/\partial n$ is prescribed on C_0 its λ-derivative is zero, and we obtain finally (for $\lambda = 0$)

$$\frac{\partial}{\partial \lambda} J_\infty[\phi] = \iint_\infty \Phi \operatorname{div}(\rho\nabla\phi)\, dx\,dy.$$

Hence an extremal ϕ must give $\operatorname{div}(\rho\nabla\phi) = 0$ at all points outside C_0, and for the infinite region we have validated the replacement of the

hydrodynamic flow-problem by the variational problem of making $J_\infty[\phi]$ stationary.

It will be observed that, in the convergence discussion, the remainders are all higher in order, by one unit, than is needed. On this account the discussion can be extended to the circulatory case, where ϕ includes a term $\kappa \log r$ in which κ is assigned, and χ is as before.

5. An example of approximation by variational procedure

We shall consider that case of the 'aerofoil problem' of section 4 in which the internal boundary C_0 is a circle, which we can take to be $r = 1$. This is a useful choice of problem for comparative purposes, since it has been attacked a number of times, by different methods.

The function $J_\infty[\phi]$ which is to be varied is given by (23), with K defined in (22). It is convenient to transform it as follows. Since $r^{-1}\cos\theta$ is harmonic, and for $r \sim \infty$

$$\chi\frac{\partial}{\partial r}\left(\frac{\cos\theta}{r}\right) = O\left(\frac{1}{r^3}\right),$$

and since C_0 is the circle $r = 1$, we have

$$\iint_\infty \nabla\frac{\cos\theta}{r}\cdot\nabla\chi\,dx\,dy = \int_{C_0} \chi\frac{\partial}{\partial n}\left(\frac{\cos\theta}{r}\right)ds = \int_{C_0} \chi\cos\theta\,ds. \qquad (25)$$

Moreover in the expression (22) for K, $\partial\phi/\partial n = 0$ since C_0 is a rigid boundary, and $\partial(Vx)/\partial n = -\partial(Vx)/\partial r = -V\cos\theta$. Hence K is equal to $V\rho_\infty$ times the integral on the left of (25), and

$$J_\infty[\phi] = \int_0^{2\pi} d\theta \int_1^\infty \left\{p - p_\infty + V\rho_\infty \nabla\left(r\cos\theta + \frac{\cos\theta}{r}\right)\cdot\nabla\chi\right\}r\,dr. \qquad (26)$$

Here p is the function of q^2 defined in (19), where $\mathbf{q} = \nabla(Vx+\chi)$. Putting

$$\chi = V\chi', \qquad \frac{V^2}{2\beta c_0^2 - V^2} = \frac{M^2}{2\beta} = \frac{(\gamma-1)M^2}{2}, \qquad \alpha = \frac{\gamma}{\gamma-1} = \beta\gamma, \quad (27)$$

so that M is the Mach number of the stream at infinity, an easy reduction gives

$$J = J_\infty[\phi] = p_\infty \int_0^{2\pi} d\theta \int_1^\infty \left[\left\{1 - (\gamma-1)M^2\left(\frac{\partial\chi'}{\partial x} + \tfrac{1}{2}(\nabla\chi')^2\right)\right\}^\alpha - 1 + \right.$$

$$\left. + \gamma M^2\left(\left(1 - \frac{\cos 2\theta}{r^2}\right)\frac{\partial\chi'}{\partial x} - \frac{\sin 2\theta}{r^2}\frac{\partial\chi'}{\partial y}\right)\right]r\,dr. \qquad (28)$$

The boundary condition on $r = 1$ is $\partial(Vx+V\chi')/\partial r = 0$, which is equivalent to

$$\frac{\partial}{\partial r}\left(\chi' - \frac{\cos\theta}{r}\right) = 0.$$

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Having regard to the symmetries of the proposed flow, and to the condition $\chi' = O(r^{-1})$ for $r \sim \infty$, we write therefore†

$$\left.\begin{aligned} \chi' &= \frac{\cos\theta}{r} + f_1(\theta)\left(\frac{1}{r} - \frac{1}{3r^3}\right) + f_2(\theta)\left(\frac{1}{3r^3} - \frac{1}{5r^5}\right) + \cdots \\ f_1(\theta) &= A\cos\theta + B\cos 3\theta + E\cos 5\theta + \ldots \\ f_2(\theta) &= C\cos\theta + D\cos 3\theta + F\cos 5\theta + \ldots \end{aligned}\right\} \tag{29}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

and propose to make J stationary by proper choice of the constants $A, B, \ldots$. For this it is necessary to have the integrand of J in a form which is explicitly and conveniently integrable while the constants remain undetermined; this is the essence of the Rayleigh–Ritz procedure. When α is not integral, as is the case for general values of γ, this will require that we approximate to $(1-\tau)^\alpha$ by means of a polynomial in τ, where

$$\tau = (\gamma - 1)M^2\left(\frac{\partial\chi'}{\partial x} + \tfrac{1}{2}(\nabla\chi')^2\right); \tag{30}$$

and it is easy to do this with adequate accuracy. We can use a curtailment of the binomial expansion

$$(1-\tau)^\alpha = 1 - \gamma M^2\left(\frac{\partial\chi'}{\partial x} + \tfrac{1}{2}(\nabla\chi')^2\right) + \frac{\alpha(\alpha-1)}{2}\tau^2 + \ldots, \tag{31}$$

or we can fit a polynomial to $(1-\tau)^\alpha$ by least-squares procedure, say

$$(1-\tau)^\alpha \approx 1 - \alpha\tau + a_2\tau^2 + a_3\tau^3 + a_4\tau^4; \tag{32}$$

here the coefficient of τ must be taken as $-\alpha$ precisely, in order that (cf. (31)) the term $\gamma M^2 \partial\chi'/\partial x$ may be cancelled from (28): for otherwise the infinite integral in (28) would not converge.

If we take $\gamma = 1\cdot405$ (a value appropriate to air), then $\alpha = 3\cdot46914$, and the least-squares fit for (32) over the range $0 \leqslant \tau \leqslant 1$ is got when

$$a_2 = 4\cdot30061, \qquad a_3 = -2\cdot17451, \qquad a_4 = 0\cdot34225;$$

the corresponding binomial coefficients are $4\cdot27, -2\cdot07, 0\cdot24$. The accuracy so obtained is indicated in the following table.

τ	$(1-\tau)^\alpha$	Percentage error in (32)
0	1	0
$\tfrac{1}{8}$	0·62925	0·030
0·16840	0·52746	0·036
$\tfrac{1}{4}$	0·36864	0·062
$\tfrac{1}{2}$	0·090303	0·11
$\tfrac{3}{4}$	0·0081544	0·26

† For the actual velocity potential, the absence of even multiples of θ from (29) implies the absence of even powers of r; so we make a similar restriction in all the functions of the field.

Here $\tau = 0.16840$ corresponds to the sonic condition $q = c$. It is clear that, for subsonic flows at least, the errors in such approximations should not outweigh those that are inherent in the use of formulae like (29) when the number of involved constants is practicably small.

The numerical work has been carried through for the case

$$M = 0.4, \qquad \gamma = 1.405,$$

using the form (29) for χ', as far as it is there explicit, along with the quartic approximation (32); actually, the contribution from the cubic and quartic terms is practically negligible, on account of the smallness of the coefficient $(\gamma - 1)M^2 = 0.0648$. $J[\phi]$ is obtained as a polynomial in A, B,..., F; and the conditions $\partial J / \partial A = 0,...$ for its maximization are a set of non-linear equations, in which, however, the linear terms are dominant, so that they can be solved iteratively. These equations are shown in Table 4.

To get an idea of the accuracy of the full solution, which will be called the third approximation (S_3), a first approximation (S_1) was found by taking $B = D = E = F = 0$, and a second (S_2) by taking $E = F = 0$. These, of course, are three distinct Rayleigh–Ritz approximations; each is found by 'strict' solution of the appropriate maximizing equations. From each of the three velocity potentials thus obtained (given in Table 1 below), the fluid speed was calculated at a set of points on or near the surface $r = 1$ of the cylinder and on the transverse axis $\theta = \frac{1}{2}\pi$. Then, for the speed at each of these points, a fourth approximation (S_4) was deduced from the previous three by applying a formula of Isakson (6). This is a formula for estimating the limit of a sequence of iterates

$$x = x_3 - \frac{(x_3 - x_2)^2}{x_3 - 2x_2 + x_1}.$$

It is here applied out of context, but with a certain plausibility, and the suggestion that emerges is that our results are correct to about 1 part in 200 and, indeed, are decidedly better at points remote from the boundary $r = 1$. These results are shown in Table 2. We show also (Table 3) a comparison with results found by Wang (variational method, $\gamma = 2$), Imai (Rayleigh–Janzen, $\gamma = 1.405$), Taylor and Sharman (electrolytic tank, $\gamma = 1.4$); and by a 'linearized solution' which is obtained from our equations (Table 4) by retaining only the linear terms. These last results are curiously close to those of Imai.

On our best approximation the whole flow-field comes out to be just subsonic. For the greatest speed q, which of course occurs at $r = 1$, $\theta = \frac{1}{2}\pi$ we find $q/V = 2.31$ or 2.32, corresponding to Mach number 0.99 or 0.995.

TABLE 1

Approximations to the velocity potential

First approximation (S_1)	$\dfrac{\phi}{V} = \left(r + \dfrac{1}{r}\right)\cos\theta + 0 \cdot 2328 \cos\theta\left(\dfrac{1}{r} - \dfrac{1}{3r^3}\right) - 0 \cdot 2196 \cos\theta\left(\dfrac{1}{3r^3} - \dfrac{1}{5r^5}\right)$
Second approximation (S_2)	$\dfrac{\phi}{V} = \left(r + \dfrac{1}{r}\right)\cos\theta + (0 \cdot 2418 \cos\theta - 0 \cdot 0551 \cos 3\theta)\left(\dfrac{1}{r} - \dfrac{1}{3r^3}\right) -$ $- (0 \cdot 1144 \cos\theta + 0 \cdot 0463 \cos 3\theta)\left(\dfrac{1}{3r^3} - \dfrac{1}{5r^5}\right)$
Third approximation (S_3)	$\dfrac{\phi}{V} = \left(r + \dfrac{1}{r}\right)\cos\theta + (0 \cdot 2428 \cos\theta - 0 \cdot 0553 \cos 3\theta + 0 \cdot 0022 \cos 5\theta)\left(\dfrac{1}{r} - \dfrac{1}{3r^3}\right) -$ $- (0 \cdot 1127 \cos\theta + 0 \cdot 0594 \cos 3\theta - 0 \cdot 0325 \cos 5\theta)\left(\dfrac{1}{3r^3} - \dfrac{1}{5r^5}\right)$

TABLE 2

Approximations to the velocity field

The tabulated numbers are q/V

	Approximation	$\theta = 10°$	$20°$	$30°$	$40°$	$50°$	$60°$	$70°$	$80°$	$90°$
$r = 1$	S_1	0·3693	0·7271	1·0630	1·3665	1·6285	1·8410	1·9975	2·0936	2·1259
	S_2	0·3084	0·6224	0·9595	1·2680	1·5795	1·8584	2·1076	2·2248	2·2747
	S_3	0·3280	0·6464	0·9536	1·2537	1·5568	1·8340	2·1074	2·2492	2·3102
	S_4	0·3227	0·6426	0·9533	1·2513	1·5372	1·8483	2·1074	2·2548	2·3214
$r = 1·04$	S_1						1·7718	1·9224	2·0145	2·0441
	S_2						1·7883	2·0274	2·1401	2·1880
	S_3						1·7649	2·0273	2·1633	2·2219
	S_4						1·7786	2·0273	2·1686	2·2323

	Approximation	$r = 1$	$1·04$	$1·2$	$1·5$	$2·0$	$3·0$
$\theta = \frac{1}{2}\pi$	S_1	2·1259	2·0441	1·7980	1·5220	1·2995	1·1348
	S_2	2·2747	2·1880	1·9139	1·5998	1·3437	1·1540
	S_3	2·3102	2·2219	1·9397	1·6144	1·3502	1·1561
	S_4	2·3214	2·2323	1·9471	1·6178	1·3513	1·1564

TABLE 3

Comparison with previous calculations of q/V

	θ	Our S_3 ($\gamma = 1\cdot405$)	Imai ($\gamma = 1\cdot405$)	Wang ($\gamma = 2$)	Linearized solution ($\gamma = 1\cdot405$)
	$10°$	0·3280	0·323	0·319	0·319
	$20°$	0·6464	0·644	0·635	0·635
	$30°$	0·9536	0·959	0·941	0·957
	$40°$	1·2537	1·266	1·247	1·260
$r = 1$	$50°$	1·5568	1·561	1·552	1·571
	$60°$	1·8340	1·836	1·845	1·836
	$70°$	2·1074	2·068	2·097	2·070
	$80°$	2·2492	2·227	2·271	2·229
	$90°$	2·3102	2·284	2·335	2·285

	r	Our S_3 ($\gamma = 1\cdot405$)	Imai ($\gamma = 1\cdot405$)	Taylor and Sharman ($\gamma = 1\cdot4$)	Wang ($\gamma = 2$)	Linearized solution ($\gamma = 1\cdot405$)
	1	2·3102	2·284	—	2·335	2·285
	1·04	2·2219	2·197	2·13	2·244	2·196
$\theta = \frac{1}{2}\pi$	1·2	1·9397	1·920	1·89	1·959	1·921
	1·5	1·6144	1·603	1·60	1·629	1·604
	2·0	1·3502	1·345	1·35	1·360	1·345
	3·0	1·1561	1·155	1·18	1·162	1·155

TABLE 4

Equations for determining the constants in formula (29)

$0\cdot837209A + 0\cdot264068B + 0\cdot136226C + 0\cdot0334949D + 0\cdot0104468E + 0\cdot00165423F = 0\cdot164996 + K_1$

$0\cdot264068A + 5\cdot47844B + 0\cdot0448888C + 0\cdot633477D + 1\cdot23090E + 0\cdot162746F = -0\cdot0258944 + K_2$

$0\cdot136226A + 0\cdot0448888B + 0\cdot0406514C + 0\cdot00710456D + 0\cdot00161371E + 0\cdot000336713F = 0\cdot0246100 + K_3$

$0\cdot0334949A + 0\cdot633477B + 0\cdot00710456C + 0\cdot110809D + 0\cdot167463E + 0\cdot0289193F = -0\cdot0306278 + K_4$

$0\cdot0104468A + 1\cdot23090B + 0\cdot00161371C + 0\cdot167463D + 14\cdot6657E + 1\cdot62026F = -0\cdot00642793 + K_5$

$0\cdot00165423A + 0\cdot162746B + 0\cdot000336713C + 0\cdot0289193D + 1\cdot62026E + 0\cdot251521F = -0\cdot00108907 + K_6$

where $K_1, K_2,..., K_6$ are polynomials in $A, B,..., F$ comprising terms of degrees 2, 3,..., 7.

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THE CHARACTER OF THE EQUILIBRIUM OF A HEAVY, VISCOUS, INCOMPRESSIBLE, ROTATING FLUID OF VARIABLE DENSITY

I. GENERAL THEORY

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SUMMARY

The equilibrium of a plane horizontal layer of a viscous incompressible fluid of variable density ρ_0 in the vertical (z) direction, which rotates uniformly at Ω rad./sec. about an axis making an angle θ with the vertical, is examined by the usual method of studying the initial behaviour of a small disturbance. Diffusion effects are ignored. The theory is developed for any general $\rho_0(z)$ and $\mu_0(z)$, where μ_0 is the coefficient of viscosity, assumed to depend only on the density.

It is shown that the solution is characterized by a variational principle in two cases, namely, (*a*) when $\theta = 0$, and (*b*) if $\theta \neq 0$, when the motion is confined to planes perpendicular to the horizontal component of Ω.

The possibility of 'overstability' (instability setting in as oscillatory motion) is discussed. A necessary, though not sufficient, condition is that $d^2\mu_0/dz^2 < 0$.

Based on the variational principle elucidated in this paper, treatments of two special density configurations, $\rho_0(z)$, are given in a following paper.

1. Introduction

THIS paper is devoted to the determination of the initial manner of development of an infinitesimal disturbance of a plane horizontal layer of a viscous incompressible heavy fluid of variable density ρ_0 in the z direction (the z axis being the upward drawn vertical) which rotates uniformly at Ω rad./sec. about an axis making an angle θ with the vertical. The variable density may be due to variable temperature or composition. The coefficient of viscosity μ_0 is taken to depend only on the density. Diffusion effects, which tend to produce changes in the density of an individual fluid particle in the course of its motion, are ignored.

The hydrodynamical flow resulting from the disturbance will tend either to restore the equilibrium state or to produce a permanent departure from it according as the density configuration $\rho_0(z)$ in the field of gravity is stable or unstable. In the stable case, gravity waves are generated. In the unstable

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case, under most conditions (see section 3 below) the system departs aperiodically from equilibrium at a rate which depends on the total horizontal wave number k of the disturbance. In general, there is one mode for which the instability is a maximum, and the properties of this mode (growth rate and wave number) are sought since during the initial period at least, it is this mode that would be observed.

The flow will be influenced by viscosity and by Coriolis forces due to rotation, to an extent determined by k. The direct damping influence of viscosity will be more important on modes of high k than on modes of low k. Coriolis forces also tend to inhibit the motion, but in a manner that is more subtle than that in which viscosity acts. They cannot dissipate energy, but in modifying the velocity field by introducing extra dynamical constraints, they enable viscosity to produce further effects which would not arise in the absence of rotation. In many hydrodynamical problems of astrophysical and geophysical interest Coriolis forces play an important role, so that any insight into the behaviour of rotating fluids that might be gained from the present study should be valuable.

Rayleigh (1) has treated the non-rotating inviscid case of the present problem. After developing a general theory for any $\rho_0(z)$ he discussed two special density configurations corresponding to (a) two superposed fluids, and (b) a continuously stratified fluid. Rayleigh's treatment of inviscid superposed fluids was extended by Bjerknes et al. (2) to include the influence of rotation. One feature worthy of note is that in the case for which the equilibrium is unstable (the heavier fluid on top), the mode of maximum instability is uninfluenced by rotation in the absence of viscosity, and always has zero wavelength.

In a recent paper Chandrasekhar (3) has extended Rayleigh's treatment of the general theory of the non-rotating case to include viscosity. An exact numerical solution to the problem of two superposed fluids of great depth was obtained for the case in which each fluid has the same kinematical viscosity coefficient. Hide (4) has made use of a variational principle enunciated by Chandrasekhar to obtain explicit, though approximate, solutions to the viscous problems corresponding to the two special density configurations introduced by Rayleigh.

In this paper the theory is developed for any $\rho_0(z)$, $\mu_0(z)$, and θ. It is shown that the solution can be characterized by a variational principle provided that either $\theta = 0$ or, in the case $\theta \neq 0$, the motion is restricted to rolls with axes parallel to the horizontal component of $\boldsymbol{\Omega}$.

Based on the theory developed here, the two special cases of a continuously stratified fluid and of two superposed fluids are treated in another paper in this journal (5).

2. The equations of the problem

The equation of relative motion appropriate to the problem is

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial}{\partial x_j} u_i - 2\rho \Omega \epsilon_{ijk} u_j v_k = \frac{\partial p_{ji}}{\partial x_j} - g\rho\lambda_i, \tag{1}$$

where the fluid is supposed to rotate uniformly about an axis whose direction is specified by the unit vector $\mathbf{v}$; and in order to use the tensor notation and the summation convention the unit vector $\boldsymbol{\lambda}$ which is in the direction of the vertical is introduced. In equation (1) ρ denotes the density, g the acceleration of gravity, Ω the angular velocity of rotation, and u_i the ith component of the (Eulerian) velocity vector; p_{ji} is the stress tensor defined by the equations

$$p_{ii} = -p + 2\mu \frac{\partial u_i}{\partial x_i}, \qquad p_{ij} = p_{ji} = \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right), \tag{2}$$

where p is the pressure and μ the coefficient of viscosity. Both μ and ρ are variable.

For an incompressible fluid the continuity condition without approximation is

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{3}$$

Because in this analysis diffusion effects are ignored, an individual particle of fluid retains the same density throughout the motion. Hence, we require that

$$\frac{D\rho}{Dt} \equiv \frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} = 0. \tag{4}$$

As we are assuming that μ is a function of ρ only, an expression similar to equation (4) must be satisfied by μ, namely

$$\frac{\partial \mu}{\partial t} + u_j \frac{\partial}{\partial x_j} \mu = 0. \tag{5}$$

Because the equilibrium situation under discussion is a static one, it is characterized by $u_i = 0$. Initially

$$\rho = \rho_0(z) \quad \text{and} \quad \mu = \mu_0(z) \tag{6}$$

and the pressure p_0 is given by the familiar hydrostatic equation

$$\frac{dp_0}{dz} + g\rho_0 = 0. \tag{7}$$

We now consider the static situation to be slightly disturbed, so that $u_i \neq 0$. We shall write

$$\rho = \rho_0(z) + \delta\rho(x,y,z,t), \qquad p = p_0(z) + \delta p(x,y,z,t) \left. \right\}$$
$$\text{and} \qquad \mu = \mu_0(z) + \delta\mu(x,y,z,t) \tag{8}$$

and treat u_i, $\delta\rho$, δp, and $\delta\mu$ as quantities of the first order of smallness so that products of such quantities can be ignored without serious error. It is convenient at this stage to introduce Cartesian coordinate axes. We take the z axis in the direction of the (upward) vertical and the x axis to be such that the (x, z) plane contains the angular velocity vector $\mathbf{\Omega}$ which then has components $(\Omega_x, 0, \Omega_z)$. If (u, v, w) are the components of $\mathbf{u}$, then on substituting in equations (1), (2), (3), and (4) we find that

$$\rho_0 \frac{\partial u}{\partial t} - 2\rho_0 \Omega_z v = -\frac{\partial \delta p}{\partial x} + \mu_0 \nabla^2 u + \left(\frac{d\mu_0}{dz}\right)\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \tag{9}$$

$$\rho_0 \frac{\partial v}{\partial t} - 2\rho_0 \Omega_x w + 2\rho_0 \Omega_z u = -\frac{\partial \delta p}{\partial y} + \mu_0 \nabla^2 v + \left(\frac{d\mu_0}{dz}\right)\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \tag{10}$$

$$\rho_0 \frac{\partial w}{\partial t} + 2\rho_0 \Omega_x v = -\frac{\partial \delta p}{\partial z} - g\,\delta\rho + \mu_0 \nabla^2 w + 2\left(\frac{d\mu_0}{dz}\right)\left(\frac{\partial w}{\partial z}\right), \tag{11}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{12}$$

and

$$\frac{\partial \delta\rho}{\partial t} + w\frac{d\rho_0}{dz} = 0. \tag{13}$$

To obtain the foregoing equations the familiar Rayleigh–Boussinesq approximation, that the variation of ρ can be ignored in all terms save the one representing the buoyancy force, is employed. This is, of course, quite consistent with the procedure of neglecting quantities of the second order of smallness.

It will be observed that a complete set of equations is obtained without having to make use of equation (5). This is because the unperturbed state here considered is static so that the variations $\delta\mu$ in μ do not enter the first order problem.

Following the usual practice in problems of this kind we seek solutions of the equations which are of the form

$$u, v, w, \delta\rho, \delta p = (\text{some function of } z) \times \exp(ik_x x + ik_y y + nt), \tag{14}$$

where k_x and k_y are the horizontal wave numbers of the harmonic perturbation; the study of a disturbance of this form can be easily generalized to more complicated cases by virtue of Fourier's theorem. On substituting for u, v, w, etc., equations (9) to (13) become

$$\rho_0 nu - 2\Omega\rho_0 v \cos\theta = -ik_x \delta p + \mu_0 (D^2 - k^2)u + (D\mu_0)(ik_x w + Du), \tag{15}$$

$$\rho_0 nv - 2\Omega\rho_0(w\sin\theta - u\cos\theta) = -ik_y \delta p + \mu_0(D^2 - k^2)v + (D\mu_0)(ik_y w + Dv), \tag{16}$$

$$\rho_0 nw + 2\Omega\rho_0 v \sin\theta = -D\delta p - g\,\delta\rho + \mu_0(D^2 - k^2)w + 2(D\mu_0)(Dw), \tag{17}$$

$$ik_x u + ik_y v + Dw = 0, \tag{18}$$

$$n\,\delta\rho + wD\rho_0 = 0, \tag{19}$$

where u, v, w, etc., now denote the dependence on z only. For brevity we have written $D = (d/dz)$, $$k^2 = k_x^2 + k_y^2 \qquad (20)$$

and θ is the angle made by the vector $\boldsymbol{\Omega}$ with the z axis.

Boundary conditions. If the fluid be confined between two rigid horizontal surfaces, then the normal component of $\mathbf{u}$ must vanish on them. On a free surface this condition is not required, but, as will appear below (section 3), it seems to be of mathematical interest. It is a convenient assumption to make when we are interested in the physical effects of density fluctuations within a fluid and wish to avoid the complications due to the density change at a free surface. In so doing we are only following Rayleigh in his successful theoretical treatment of the now classical Bénard convection problem. Thus

$$w = 0 \quad \text{on a rigid or free surface.} \qquad \text{(i)}$$

At the physical interface between two fluids in contact with each other, the pressure and the normal components of velocity must be continuous. Thus we must require that

$$p_1 = p_2 \quad \text{at the physical interface,} \qquad \text{(ii)}$$

where the subscripts 1 and 2 refer, respectively, to the lower and upper fluids, and it can be shown that to the approximation of the present theory

$$w_1 = w_2 \quad \text{at the level of the undisturbed interface.} \qquad \text{(iii)}$$

Extra conditions are imposed by viscosity. We shall anticipate here the necessity of introducing the z component ζ of the vorticity vector,

$$\boldsymbol{\omega} = \operatorname{curl} \mathbf{u} = (\xi, \eta, \zeta), \qquad (21)$$

so that $$\zeta(z) = ik_x v(z) - ik_y u(z). \qquad (22)$$

Because both u and v must vanish on a rigid horizontal surface, by equations (22) and (18) we must require that

$$\zeta = Dw = 0 \quad \text{(rigid surface).} \qquad \text{(iv a)}$$

If the surface is free then the tangential viscous stress there must vanish. To the approximation of the first order theory, this condition would be satisfied if $$D^2w = D\zeta = 0 \quad \text{(free surface).} \qquad \text{(iv b)}$$

In deriving (iv b) use has been made of equation (2) and condition (i).

At the interface between two fluids in contact, viscosity demands that there be no discontinuity in the tangential velocity components and in the viscous stresses, thus giving rise to the following conditions:

$$Dw_1 = Dw_2 \quad \text{and} \quad \zeta_1 = \zeta_2, \qquad \text{(v)}$$

and $$\mu_1(D^2+k^2)w_1 = \mu_2(D^2+k^2)w_2 \qquad \text{(vi)}$$

at the undisturbed level of the interface.

Equations (15) to (19) and boundary conditions (i) to (vi) completely determine the solution.

On multiplying equations (15) and (16) by ik_x and ik_y respectively and adding, we obtain on making further use of equations (20), (18), and (22),

$$k^2\delta p = -\rho_0 nDw - 2\Omega\rho_0(\zeta\cos\theta + ik_y w\sin\theta) +$$
$$+ \mu_0(D^2 - k^2)Dw + (D\mu_0)(D^2 + k^2)w. \quad (23)$$

When v and $\delta\rho$ have been eliminated from equation (23) with the aid of equations (22), (18), and (19) we find that

$$k^2 D\delta p = -\rho_0 nk^2 w + 2\Omega\rho_0\sin\theta(ik_y Dw - ik_x\zeta) +$$
$$+ \mu_0 k^2(D^2 - k^2)w + 2k^2(D\mu_0)Dw + g(k^2/n)(D\rho_0)w. \quad (24)$$

The result of eliminating δp between equations (23) and (24) gives the equation

$$[\rho_0 nk^2 - g(k^2/n)D\rho_0]w - nD(\rho_0 Dw) + \mu_0(D^2 - k^2)^2 w +$$
$$+ 2(D\mu_0)(D^2 - k^2)Dw + (D^2\mu_0)(D^2 + k^2)w$$
$$= 2\Omega[D(\rho_0\zeta)\cos\theta + i(k_y(D\rho_0)w + \rho_0 k_x\zeta)\sin\theta]. \quad (25)$$

We require one further equation, which is to be obtained by eliminating δp between equations (15) and (16). Thus, with the aid of equations (18) and (22) we obtain the equation

$$\rho_0 n\zeta - \mu_0(D^2 - k^2)\zeta - D\mu_0 D\zeta = 2\Omega\rho_0(D\cos\theta + ik_x\sin\theta)w. \quad (26)$$

We shall not write down the sixth order differential equation which results from eliminating ζ between the last two equations, but rather deal with them as a pair.

3. A variational principle

Let there be solutions w_i and w_j which belong, respectively, to the two characteristic values n_i and n_j of equations (25) and (26). Equation (25) is equivalent to the two equations (23) and (24). Multiply equation (23) for w_i by $-Dw_j$ and equation (24) for w_i by w_j and integrate the resulting equations with respect to z over the whole vertical range covered by the fluid, making use of the fact that on integrating by parts

$$\int_L w_j D\delta p_i \, dz = -\int_L \delta p_i Dw_j \, dz \quad (27)$$

(where L signifies that we are integrating over the whole system). Because the integrated part vanishes by virtue of boundary condition (i), we can

then eliminate δp_i and are left with the one equation

$$n_i\left\{\int_L \rho_0 w_i w_j\, dz - \frac{1}{k^2}\int_L w_j D(\rho_0 Dw_i)\, dz\right\} - \frac{g}{n_i}\int_L (D\rho_0)w_i w_j\, dz +$$
$$+ \int_L \mu_0[k^2 w_i w_j - w_j D^2 w_i + (Dw_i)(Dw_j)]\, dz -$$
$$- \int_L \left[(D\mu_0)(w_i Dw_j + 2w_j Dw_i) + \frac{1}{k^2}Dw_j D(\mu_0 D^2 w_i)\right] dz +$$
$$+ \frac{2\Omega}{k^2}\left\{\cos\theta \int_L \rho_0 \zeta_i Dw_j\, dz + i\sin\theta \int_L \rho_0[k_y D(w_i w_j) - k_x w_j \zeta_i]\, dz\right\} = 0.$$

(28)

Multiply equation (26) for w_j by ζ_i and integrate. Then, after some rearranging, we find that

$$2\Omega\cos\theta \int_L \rho_0 \zeta_i Dw_j\, dz = \int_L (\rho_0 n_j + \mu_0 k^2)\zeta_i \zeta_j\, dz -$$
$$- \int_L \zeta_i D(\mu_0 D\zeta_j)\, dz - 2\Omega i k_x \sin\theta \int_L \rho_0 w_j \zeta_i\, dz. \quad (29)$$

We can now replace the fifth term of equation (28) by the right-hand side of equation (29), and thereby deduce that

$$n_i\left\{k^2 \int_L \rho_0 w_i w_j\, dz - \int_L w_j D(\rho_0 Dw_i)\, dz\right\} - \frac{gk^2}{n_i}\int_L (D\rho_0)w_i w_j\, dz +$$
$$+ \int_L (n_j \rho_0 + \mu_0 k^2)\zeta_i \zeta_j\, dz + k^2 \int_L \mu_0 w_j(k^2 - D^2)w_i\, dz - \int_L \zeta_i D(\mu_0 D\zeta_j)\, dz -$$
$$- k^2 \int_L (D\mu_0)(w_i Dw_j + 2w_j Dw_i)\, dz + \int_L (Dw_j)(\mu_0 k^2 Dw_i - D(\mu_0 D^2 w_i))\, dz +$$
$$+ 2\Omega i \sin\theta \int_L \rho_0(k_y D(w_i w_j) - 2k_x w_j \zeta_i)\, dz = 0. \quad (30)$$

We can simplify equation (30) by noting that on integration by parts

$$\int_L w_j D(\rho_0 Dw_i)\, dz = -\int_L \rho_0(Dw_i)(Dw_j)\, dz \quad (31)$$

by virtue of boundary condition (i),

$$\int_L \zeta_i D(\mu_0 D\zeta_j)\, dz = -\int_L \mu_0(D\zeta_i)(D\zeta_j)\, dz \quad (32)$$

and $$\int_L (Dw_j)D(\mu_0 D^2 w_i)\, dz = -\int_L \mu_0(D^2 w_i)(D^2 w_j)\, dz \quad (33)$$

by virtue of boundary condition (iv), and

$$\int_L \{\mu_0 w_j D^2 w_i + (D\mu_0)(w_i Dw_j + 2w_j Dw_i)\}\, dz$$
$$= -\int_L \{\mu_0(Dw_i)(Dw_j) + (D^2\mu_0)w_i w_j\}\, dz \quad (34)$$

by virtue of boundary conditions (i) and (iv). Finally, on introducing equations (31)–(34) into equation (30) we obtain the equation

$$n_i I_1(i,j) - (g/n_i)I_2(i,j) + I_3(i,j) + n_j I_4(i,j) + I_5(i,j) + I_6(i,j) = 0, \quad (35)$$

where for brevity we have written

$$I_1(i,j) = \int_L \rho_0\left\{w_i w_j + \frac{1}{k^2}(Dw_i)(Dw_j)\right\} dz, \tag{36}$$

$$I_2(i,j) = \int_L (D\rho_0)w_i w_j\, dz, \tag{37}$$

$$I_3(i,j) = \int_L \mu_0\left\{k^2 w_i w_j + 2(Dw_i)(Dw_j) + \frac{1}{k^2}(D^2 w_i)(D^2 w_j)\right\} dz +$$
$$+ \int_L (D^2\mu_0)w_i w_j\, dz, \tag{38}$$

$$I_4(i,j) = \frac{1}{k^2}\int_L \rho_0 \zeta_i \zeta_j\, dz, \tag{39}$$

$$I_5(i,j) = \int_L \mu_0\left\{\zeta_i \zeta_j + \frac{1}{k^2}(D\zeta_i)(D\zeta_j)\right\} dz, \tag{40}$$

and

$$I_6(i,j) = \frac{2\Omega i \sin\theta}{k^2}\int_L \rho_0\{k_y D(w_i w_j) - 2k_x w_j \zeta_i\}\, dz. \tag{41}$$

It will be observed that

$$I_s(i,j) = I_s(j,i), \qquad 1 \leqslant s \leqslant 5 \tag{42}$$

and

$$I_6(i,j) = I_6(j,i) + I_7(i,j), \text{ say,} \tag{43}$$

where

$$I_7(i,j) = \frac{4\Omega i k_x \sin\theta}{k^2}\int_L \rho_0(w_i \zeta_j - w_j \zeta_i)\, dz. \tag{44}$$

Let us define one more integral, namely

$$I_8(i,j) = \frac{4\Omega i \sin\theta}{k^2}\int_L \rho_0\{k_y D(w_i w_j) - k_x(\zeta_i w_j + \zeta_j w_i)\}\, dz \tag{45}$$

so that

$$I_6(i,j) + I_6(j,i) = I_8(i,j) = I_8(j,i). \tag{46}$$

On interchanging the subscripts i and j in equation (35) and then adding the resulting equation to it, we find that

$$(n_i + n_j)\{I_1(i,j) + I_4(i,j) - (g/(n_i n_j))I_2(i,j)\} + 2\{I_3(i,j) + I_5(i,j)\} + I_8(i,j) = 0. \tag{47}$$

If instead of adding we subtract, the result

$$(n_i - n_j)\{I_1(i,j) - I_4(i,j) + (g/(n_i n_j))I_2(i,j)\} + I_7(i,j) = 0 \qquad (48)$$

is obtained. In deriving the two foregoing equations, use has been made of equations (32)–(36).

Now put $i = j$ in equation (47), so that $n_i = n_j = n$ (say) where

$$n(I_1 + I_4) - (g/n)I_2 + I_3 + I_5 + \tfrac{1}{2}I_8 = 0 \qquad (49)$$

and (see equations (32)–(36))

$$I_1 = \int_L \rho_0 \left\{ w^2 + \frac{1}{k^2}(Dw)^2 \right\} dz, \qquad (50)$$

$$I_2 = \int_L (D\rho_0)w^2 \, dz, \qquad (51)$$

$$I_3 = \int_L \mu_0 \left\{ k^2 w^2 + 2(Dw)^2 + \frac{1}{k^2}(D^2 w)^2 \right\} dz + \int_L (D^2\mu_0)w^2 \, dz, \qquad (52)$$

$$I_4 = \frac{1}{k^2} \int_L \rho_0 \zeta^2 \, dz, \qquad (53)$$

$$I_5 = \int_L \mu_0 \left\{ \zeta^2 + \frac{1}{k^2}(D\zeta)^2 \right\} dz, \qquad (54)$$

and $\qquad \tfrac{1}{2}I_8 = \dfrac{2\Omega i \sin\theta}{k^2} \displaystyle\int_L \rho_0\{k_y D(w^2) - 2k_x w\zeta\} \, dz. \qquad (55)$

Consider the variation δn in n which would be caused by small variations δw and $\delta\zeta$ in w and ζ respectively. The only other limitation which we shall impose on δw and $\delta\zeta$ is that they must satisfy the boundary conditions. Associated with these changes in w and ζ are changes δI_s in the I_s's. We can study these changes by using equation (49) from which we deduce that

$$-\delta n\{I_1 + I_4 + (g/n^2)I_2\} = n\{\delta I_1 + \delta I_4\} - (g/n)\delta I_2 + \delta I_3 + \delta I_5 + \tfrac{1}{2}\delta I_8. \qquad (56)$$

We now use equations (50)–(55) to evaluate the δI's; we find

$$\tfrac{1}{2}\delta I_1 = \int_L \rho_0 \left\{ w\,\delta w + \frac{1}{k^2}(Dw)(D\,\delta w) \right\} dz = \int_L \left\{ \rho_0 w - \frac{1}{k^2}D(\rho_0\,Dw) \right\} \delta w \, dz \qquad (57)$$

because on integrating the second term in the integrand by parts the integrated part vanishes as δw satisfies the boundary conditions. By an analogous procedure, it can be shown that

$$\tfrac{1}{2}\delta I_2 = \int_L (D\rho_0)w\,\delta w \, dz, \qquad (58)$$

$$\tfrac{1}{2}\delta I_3 = \int_L \left\{ \mu_0\left(k^2 w\,\delta w + 2(Dw)(D\,\delta w) + \frac{1}{k^2}(D^2 w)(D^2\,\delta w)\right) + (D^2\mu_0)w\,\delta w\right\}\,dz$$

$$= \int_L \left\{ \mu_0 k^2 w - 2D(\mu_0\,Dw) + \frac{1}{k^2}D^2(\mu_0\,D^2 w) + (D^2\mu_0)w\right\}\,\delta w\,dz, \quad (59)$$

$$\tfrac{1}{2}\delta I_4 = \frac{1}{k^2}\int_L \rho_0\,\zeta\,\delta\zeta\,dz, \qquad (60)$$

$$\tfrac{1}{2}\delta I_5 = \int_L \mu_0\left\{\zeta\,\delta\zeta + \frac{1}{k^2}(D\zeta)(D\,\delta\zeta)\right\}\,dz = \int_L \left\{\mu_0\zeta - \frac{1}{k^2}D(\mu_0\,D\zeta)\right\}\,\delta\zeta\,dz, \quad (61)$$

and

$$\tfrac{1}{4}\delta I_8 = \frac{2\Omega i\sin\theta}{k^2}\int_L \rho_0\{k_y(\delta w\,Dw + wD\,\delta w) - k_x(w\,\delta\zeta + \zeta\,\delta w)\}\,dz$$

$$= -\frac{2\Omega i\sin\theta}{k^2}\left[\int_L \{k_y(D\rho_0)w + \rho_0 k_x\zeta\}\,\delta w\,dz + \int_L \rho_0 k_x w\,\delta\zeta\,dz\right]. \quad (62)$$

We can now substitute for the δI's in equation (56) and obtain

$$-\frac{\delta n}{2}\left\{I_1 + I_4 + \frac{gI_2}{n^2}\right\} = \int_L \left\{n\rho_0 - \frac{g}{n}D\rho_0 + \mu_0 k^2 + D^2\mu_0\right\}w\,\delta w\,dz +$$

$$+ \int_L \left\{\frac{1}{k^2}D^2(\mu_0\,D^2 w) - 2D(\mu_0\,Dw) - \frac{n}{k^2}D(\rho_0\,Dw) - \right.$$

$$\left. - \frac{2\Omega i\sin\theta}{k^2}(k_y wD\rho_0 + \rho_0 k_x\zeta)\right\}\,\delta w\,dz +$$

$$+ \int_L \left\{\frac{n}{k^2}\rho_0\zeta + \mu_0\zeta - \frac{1}{k^2}D(\mu_0\,D\zeta) - \frac{2\Omega i\sin\theta}{k^2}\rho_0 k_x w\right\}\,\delta\zeta\,dz. \quad (63)$$

But by inspection of equations (25) and (26), it is clear that we can write equation (63) as follows:

$$-\frac{\delta n}{2}\left\{I_1 + I_4 + \frac{gI_2}{n^2}\right\} = \frac{2\Omega\cos\theta}{k^2}\int_L \{(\rho_0\,Dw)\,\delta\zeta + D(\rho_0\zeta)\,\delta w\}\,dz, \quad (64)$$

and if we integrate by parts the second term in the integrand of the right-hand side of the last equation,

$$-\frac{\delta n}{2}\left\{I_1 + I_4 + \frac{gI_2}{n^2}\right\} = \frac{2\Omega\cos\theta}{k^2}\int_L \{(\rho_0\,Dw)\,\delta\zeta - \rho_0\zeta D\,\delta w\}\,dz = I, \text{ say.} \quad (65)$$

Equation (26) becomes after slight rearranging

$$(2\Omega\rho_0\cos\theta)Dw = -D(\mu_0\,D\zeta) + (\mu_0 k^2 + \rho_0 n)\zeta - 2\Omega\rho_0 k_x wi\sin\theta, \quad (26')$$

from which it is easily deduced that

$$(2\Omega\rho_0\cos\theta)D\,\delta w$$
$$= -D(\mu_0 D\,\delta\zeta)+(\mu_0 k^2+\rho_0 n)\,\delta\zeta + \rho_0\zeta\,\delta n - 2\Omega\rho_0 ik_x\,\delta w\sin\theta. \quad (66)$$

The quantity I can now be evaluated by multiplying equation (26′) by $\delta\zeta$ and equation (66) by ζ, subtracting and integrating. Thus, we are left with the equation

$$-\frac{\delta n}{2}\Big\{I_1-I_4+\frac{gI_2}{n^2}\Big\} = \frac{2\Omega ik_x\sin\theta}{k^2}\int_L \rho_0(\zeta\,\delta w - w\,\delta\zeta)\,dz. \quad (67)$$

According to equation (67), δn is not, in general, zero. However, when $\theta = 0$, i.e. when the system rotates about a vertical axis, $\delta n = 0$ and we have a variational principle. It also follows that even when θ is not zero, if we consider a disturbance periodic in the y direction only, so that $k_x = 0$, a variational principle still applies. We might try to put a physical interpretation on the latter results. Equation (49) represents the energy equation and states that the kinetic energy of the system increases at a rate which is equal to the difference between the rates at which buoyancy forces do work on the fluid and viscous forces convert the kinetic energy of flow into heat. Thus, equation (67) states that when $\theta = 0$, for given k_x and k_y, the velocity field adjusts itself so that the rate of change of kinetic energy is a maximum. The result that when θ is not equal to zero, δn is zero only when k_x vanishes, suggests that the motion would have the form of horizontal rolls whose axes are in the planes parallel to that formed by the vectors $\mathbf{g}$ and $\mathbf{\Omega}$, if the velocity field tends to adjust itself until the kinetic energy increases at a maximum rate.

Since $\delta n = 0$ when $\theta = 0$ or $k_x = 0$ for small variations δw in w that satisfy the boundary conditions, equation (49) could be used as a basis for evaluating n by a variational procedure. One would assign an approximate trial function $w(z)$ and thereby evaluate the zeroth approximation to n which, by equation (67), should not contain a large error. This value of n would enable a better approximation to $w(z)$ to be found, which in its turn would give rise to a first approximation to n, and so on. By repeating this procedure, n could be found to any desired accuracy.

Since the principal objective of investigating problems of this type is to elucidate the essential physical features, high numerical accuracy is not a prime consideration. The zeroth approximation should be good enough in most cases (5).

Further discussion of the properties of n. When $\theta = 0$ (or $k_x = 0$) we can write

$$(n_i-n_j)\{I_1(i,j)-I_4(i,j)+(g/(n_i n_j))I_2(i,j)\} = 0 \quad (68)$$

and

$$(n_i+n_j)\{I_1(i,j)+I_4(i,j)-(g/(n_i n_j))I_2(i,j)\}+2\{I_3(i,j)+I_5(i,j)\} = 0 \quad (69)$$

in place of equations (47) and (48), because then $I_7(i,j) = I_8(i,j) = 0$.

Let there be solutions w and w^* associated with ζ and ζ^*, and n and n^*, respectively, where the star denotes the complex conjugate. Thus if

$$n = \mathrm{re}(n)+i\,\mathrm{im}(n) \quad (70)$$

then by equations (68) and (69)

$$\mathrm{im}(n)\{I_1(i,i^*)-I_4(i,i^*)+(g/|n|^2)I_2(i,i^*)\} = 0 \quad (71)$$

and

$$\mathrm{re}(n)\{I_1(i,i^*)-(g/|n|^2)I_2(i,i^*)+I_4(i,i^*)\} = -I_3(i,i^*)-I_5(i,i^*). \quad (72)$$

From the definitions of $I_s(i,j)$, $I_s(i,i^*)$ $(1 \leqslant s \leqslant 5)$ are real (see equations (36)–(40)). If $\mathrm{im}(n) \neq 0$ then by equation (71)

$$gI_2(i,i^*) = |n|^2\{I_4(i,i^*)-I_1(i,i^*)\}; \quad (73)$$

and therefore,

$$\mathrm{re}(n) = -\tfrac{1}{2}\{I_3(i,i^*)+I_5(i,i^*)\}/I_1(i,i^*) \quad (74)$$

when the imaginary part of n is not zero. Now $I_5(i,i^*)$ and $I_1(i,i^*)$ are essentially positive. $I_3(i,i^*)$ is also essentially positive if

$$\int_L \mu_0\left\{k^2|w|^2+2|Dw|^2+\frac{1}{k^2}|D^2w|^2\right\} dz > \int_L (D^2\mu_0)|w|^2\, dz \quad (75)$$

(see equation (38)), which would certainly be satisfied if

$$D^2\mu_0 > 0 \quad (76)$$

and even if $D^2\mu_0 < 0$, only when $|D^2\mu_0|$ is very large would $I_3(i,i^*)$ possibly become negative. Thus, when (75) is satisfied, from equation (74) we see that n only has a non-zero imaginary part when the real part of n is negative. Hence, all oscillatory solutions would be damped unless (75) were not satisfied, in which case the possibility of overstability (i.e. motion setting in as oscillations of increasing amplitude with time) would arise. Chandrasekhar (3) found a similar result in the non-rotating case. The physical interpretation of the result that overstability may not be excluded as a possible type of motion when $D^2\mu_0 < 0$ is by no means clear. This point requires further attention and would probably be clarified by an examination of a specific problem in which the viscosity is made to vary appropriately with z. Such a problem has not yet been attempted, but will be in the near future.

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THE CHARACTER OF THE EQUILIBRIUM OF A HEAVY, VISCOUS, INCOMPRESSIBLE, ROTATING FLUID OF VARIABLE DENSITY

II. TWO SPECIAL CASES

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SUMMARY

In a previous paper, the equilibrium of a plane horizontal layer of a viscous incompressible fluid of variable density ρ_0 in the vertical (z) direction, which rotates uniformly at Ω rad./sec. about an axis making an angle θ with the vertical, was examined by the usual method of studying the initial behaviour of a small disturbance. The general properties of the ensuing hydrodynamical motion for any $\rho_0(z)$ and $\mu_0(z)$ (μ_0 being the coefficient of viscosity) were discussed. It was shown that in some circumstances, including the case $\theta = 0$, the solution is characterized by a variational principle.

In the present paper, approximate methods suggested by the variational principle are applied to two special problems, in each of which $\theta = 0$.

The first is that of a continuously stratified fluid of finite depth in which

$$\rho_0(z) = \rho_1 \exp(\beta z), \quad \beta > 0.$$

The properties of the mode of maximum instability, characterized by its growth rate, n_m, and its total wave number, k_m, depend only on two dimensionless parameters $G = (g\beta d^4/\nu^2)$ and $T = (16\Omega^2 d^4/\pi^4\nu^2)$ if n_m and k_m are measured in suitable units, where g is the acceleration of gravity and ν is the coefficient of kinematical viscosity, assumed for simplicity to be constant. Values of n_m and k_m are calculated for many assigned values of G and T. These results show that instability is inhibited by the combined influence of viscosity and rotation. For a given rate of rotation, the inhibition is most pronounced when ν has a finite, non-zero value.

The second case examined is that of two very deep superposed fluids of density ρ_1 and ρ_2, the subscripts referring to the lower and upper fluids respectively. Again, ν is assumed to be constant, and only the unstable case ($\rho_2 > \rho_1$) is considered. The properties of the mode of maximum instability are influenced by rotation to an extent measured by the dimensionless parameter $4\Omega^3\nu(\rho_1 + \rho_2)^2/(g^2(\rho_2 - \rho_1)^2)$.

1. Introduction

A previous paper (**1**) presented the theory of the initial manner of development of an infinitesimal disturbance of a plane horizontal layer of a heavy viscous incompressible fluid of density $\rho_0(z)$ (z being the upward vertical coordinate), which rotates uniformly at Ω rad./sec. about an axis making an angle θ with the z axis. Diffusion effects were ignored. In that paper, no restriction was placed on the form of the density field $\rho_0(z)$ and of the viscosity, $\mu_0(z)$. However, attention was given only to a broad discussion of

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the general properties of the hydrodynamical flow ensuing after the initial disturbance, and to certain properties of the equations of the problem. For this reason, it was entitled 'general theory'.

Since the study of the problem under consideration was initiated for the purpose of gaining insight into the combined influence of viscosity and Coriolis forces on hydrodynamical flow produced by gravity acting on an inhomogeneous fluid, specific situations in which $\rho_0(z)$ and $\mu_0(z)$ are specified have to be considered. It is the purpose of the present paper to deal with two such problems in sections 3 and 4 respectively. The first is that of a fluid of finite depth and constant coefficient of kinematical viscosity ν, continuously stratified according to the law $\rho_0(z) = \rho_1 \exp(\beta z)$ where ρ_1 and β are constants. The second is that of two very deep superposed immiscible fluids of different density; again the simplification that ν is a constant is employed. In all cases discussed, $\theta = 0$.

The aforementioned density configurations were first introduced by Rayleigh (2).

2. Previous mathematical investigations

A general introduction to the topic under discussion in this paper and in (1) is contained in the latter paper. The development of the theory presented in (1) follows the familiar procedure for dealing with small perturbations from an equilibrium state. In order to introduce the notation and to present logically those results of (1) required in sections 2 and 3 of the present paper, we shall sketch the theory here.

In the equilibrium state, since there is no motion the pressure p has the hydrostatic value p_0 (say) corresponding to gravity (acceleration g) acting on the undisturbed density field $\rho_0(z)$.

Consider now that at time $t = 0$, the fluid is disturbed. At any time t, at a point in space with coordinates (x, y, z) (z being the upward drawn vertical and the (x, y) plane being horizontal) the pressure is no longer hydrostatic. We write

$$p = p_0(z) + \delta p(x, y, z, t) \tag{1}$$

corresponding to rearranged density and viscosity fields,

$$\rho = \rho_0(z) + \delta \rho(x, y, z, t) \tag{2}$$

and

$$\mu = \mu_0(z) + \delta \mu(x, y, z, t). \tag{3}$$

The corresponding velocity field $\mathbf{u}(x, y, z, t)$ has components (u, v, w). It is convenient to introduce the vorticity vector $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u} = (\xi, \eta, \zeta)$.

The equations governing the flow consist of the equation of hydrodynamical motion, that of continuity of mass for an incompressible fluid, and two equations stating that since diffusion effects are ignored and

depends only on ρ, each particle of fluid preserves its initial values of ρ and μ throughout its subsequent motion. By limiting our discussion to small disturbances, so that to a first approximation quantities of the second order or greater in δp, $\delta\rho$, $\delta\mu$, $\mathbf{u}$ and $\boldsymbol{\omega}$ can be ignored, the set of equations can be made linear in these variables.

The next step is to write all the variables in the form

$$(\text{function of } z \text{ only}) \times \exp(ik_x x + ik_y y + nt),$$

where k_x and k_y are the two components of the horizontal total wave number k and n is the 'growth rate' of the disturbance. Thus, we consider only disturbances with simple harmonic form in the horizontal. This procedure can easily be generalized to a more complicated form of disturbance by virtue of Fourier's theorem.

In case $\theta = 0$ it is found that w and ζ satisfy the following equations (see (1), equations 25 and 26)

$$[\rho_0 nk^2 - (gk^2/n)D\rho_0]w - nD(\rho_0 Dw) + \mu_0(D^2-k^2)^2 w +$$
$$+ 2(D\mu_0)(D^2-k^2)Dw + (D^2\mu_0)(D^2+k^2)w = 2\Omega D(\rho_0 \zeta), \quad (4)$$

and $\qquad \rho_0 n\zeta - \mu_0(D^2-k^2)\zeta - D\mu_0 D\zeta = 2\Omega\rho_0 Dw, \qquad (5)$

where w and ζ now denote the dependence on z of the vertical components of $\mathbf{u}$ and $\boldsymbol{\omega}$ respectively and D denotes (d/dz).

Any problem is entirely determined by equations (4) and (5) and the appropriate boundary conditions, which are as follows (see (1), section 2):

On a rigid surface, the normal component of $\mathbf{u}$ must vanish. On a free surface, it is often convenient to make this component vanish when problems are examined in which the potential energy due to the deformation of the surface is of no interest. Thus

(i) $w = 0$ on a rigid surface and on a free surface.

Owing to viscosity, there can be no slip at a rigid surface. This requires that

(ii a) $\zeta = Dw = 0$ on a rigid surface.

The viscous condition at a free surface is that the tangential stress component should vanish. This requires that

(ii b) $D\zeta = D^2 w = 0$ on a free surface.

Further requirements have to be imposed at interfaces between different fluids. These are that the velocity and the stresses within the fluid must

be continuous. In particular, to the order of approximation of the present theory, it can be shown that when two fluids are in contact

(iii) Dw, w, and $\mu(D^2+k^2)w$ are continuous at the undisturbed level of the interface, and

(iv) p is continuous at the disturbed interface.

It is shown in (1) that equations (4) and (5) are equivalent to the equation

$$n(I_1+I_4)-(g/n)I_2+I_3+I_5 = 0, \tag{6}$$

where

$$I_1 = \int_L \rho_0 \left\{ w^2 + \frac{1}{k^2}(Dw)^2 \right\} dz, \tag{7}$$

$$I_2 = \int_L (D\rho_0)w^2 \, dz, \tag{8}$$

$$I_3 = \int_L \mu_0 \left\{ k^2 w^2 + 2(Dw)^2 + \frac{1}{k^2}(D^2w)^2 \right\} dz + \int_L (D^2\mu_0)w^2 \, dz, \tag{9}$$

$$I_4 = \frac{1}{k^2} \int_L \rho_0 \zeta^2 \, dz, \tag{10}$$

$$I_5 = \int_L \mu_0 \left\{ \zeta^2 + \frac{1}{k^2}(D\zeta)^2 \right\} dz, \tag{11}$$

where the L signifies that the integrals must be taken over the whole vertical extent of the fluid (see (1), equation (49)). It is also shown that δn, the change in n resulting from small first order changes δw and $\delta \zeta$ in w and ζ, is zero if δw and $\delta \zeta$ satisfy the boundary conditions of the problem. Thus the solution is characterized by a variational principle.

The existence of this principle suggests approximate methods for solving specific problems, since fairly large errors in w and ζ should give rise to only small errors in n, the quantity we wish to find.

It is readily shown that if $w(z)$ does not involve n equation (6) is a cubic in n, and when $\Omega = 0$, the cubic reduces to a quadratic. However, the true value of $w(z)$ does involve n and equation (6) gives rise to an equation of degree higher than three.

In the application of equation (6) to the problems with which this paper is concerned, we shall take for $w(z)$ that value which would be exact only in the absence of viscosity. This implies that the boundary conditions arising on account of viscosity cannot be completely satisfied; they are not entirely ignored in the theory since they are employed in the derivation of equation (6).

It is a matter of experience that at least in slightly less complicated problems than those in hand, the non-viscous $w(z)$ is a good trial function to use (Chandrasekhar (3), Hide (4)).

3. The equilibrium of a continuously stratified fluid of finite depth

In this section the special case of a layer of fluid confined between two rigid horizontal boundaries located at $z = 0$ and $z = d$ is considered. The density stratification will be taken to follow the law

$$\rho_0(z) = \rho_1 e^{\beta z}, \tag{12}$$

where ρ_1 and β are constants.

An assumption which will greatly simplify the analysis without removing any essential physical feature will be made, namely, that the coefficient of kinematical viscosity ν is a constant. Hence, we shall write

$$\mu_0(z) = \nu \rho_1 e^{\beta z}. \tag{13}$$

The basic equations of our problem are equations (6) and (5). We employ equation (6) because only an approximate solution is sought. An exact treatment would require that we solve equations (4) and (5) with respect to the boundary conditions (see section 2)

$$w(0) = w(d) = 0 \tag{14}$$

and $$\zeta(0) = \zeta(d) = 0, \qquad Dw(0) = Dw(d) = 0, \tag{15}$$

where it will be recalled that the last two conditions express the 'no slip at the boundaries' requirement when the fluid is viscous.

We shall assume that

$$w(z) = A \sin(s\pi z/d), \tag{16}$$

where A is a constant, and s is an integer. If

$$\zeta(z) = B \cos(s\pi z/d) \tag{17}$$

and $$(\beta d/s\pi) \ll 1 \tag{18}$$

(so that to a first approximation we can write $1 + (\beta d/s\pi) \doteq 1$), then by equation (5)

$$B = A(2\pi\Omega s/d)/(n + \nu(k^2 + s^2\pi^2/d^2)). \tag{19}$$

The assumption expressed by the inequality (18) would be valid if the density difference between points located at neighbouring nodes in the velocity field were always much less than the average density of the fluid; this would be true in many cases of practical interest.

With the assumed form for $w(z)$, although (14) is satisfied, (15) is not. The justification of the procedure of taking a trial function which only satisfies the 'non-viscous' boundary conditions has already been discussed in section 2.

On substituting for $w(z)$ and $\zeta(z)$ in equation (6) we find that

$$n^2\rho_1 A^2 \int_0^d \left\{\sin^2\!\left(\frac{s\pi z}{d}\right)+\left(\frac{s\pi}{kd}\right)^2\cos^2\!\left(\frac{s\pi z}{d}\right)\right\}e^{\beta z}\,dz \;-\; g\rho_1\beta A^2 \int_0^d e^{\beta z}\sin^2\!\left(\frac{s\pi z}{d}\right)dz\,+$$

$$+\,n\nu\rho_1 A^2 \int_0^d \left\{\left[k^2+\beta^2+\left(\frac{s^2\pi^2}{kd^2}\right)^2\right]\sin^2\!\left(\frac{s\pi z}{d}\right)+2\left(\frac{s\pi}{d}\right)^2\cos^2\!\left(\frac{s\pi z}{d}\right)\right\}e^{\beta z}\,dz$$

$$=\;-\frac{n^2}{k^2}\rho_1 B^2 \int_0^d e^{\beta z}\cos^2\!\left(\frac{s\pi z}{d}\right)dz\,-$$

$$-\,\nu n\rho_1 B^2 \int_0^d \left\{\cos^2\!\left(\frac{s\pi z}{d}\right)+\left(\frac{s\pi}{kd}\right)^2\sin^2\!\left(\frac{s\pi z}{d}\right)\right\}e^{\beta z}\,dz. \quad (20)$$

After evaluating the integrals in this equation, and substituting for B from equation (19), we are left with the expression

$$n^2(1+s^2\pi^2/k^2d^2)+\nu k^2 n(1+s^2\pi^2/k^2d^2)^2-g\beta$$
$$=\;-4\Omega^2 s^2\pi^2 n/\{k^2d^2\big(n+\nu k^2(1+s^2\pi^2/k^2d^2)\big)\}. \quad (21)$$

In deriving the last equation, the approximation that $(\beta d/s\pi)\ll 1$ which was introduced earlier has been employed. Equation (21) reduces to equation (50) of (**2**) when $\Omega=0$.

Let us measure k and n in terms of the units (π/d) cm.$^{-1}$ and $(\pi^2\nu/2d^2)$ sec.$^{-1}$, respectively, so that we may write

$$n^2(1+s^2/k^2)+2nk^2(1+s^2/k^2)^2-4G \;=\; -Tn/\{k^2(n+2k^2+2s^2)\} \quad (22)$$

in place of equation (21), where

$$G \equiv (g\beta d^4)/(\pi^4\nu^2) \quad (23)$$

and

$$T \equiv (16\Omega^2 d^4)/(\pi^4\nu^2). \quad (24)$$

G and T are pure numbers. G is a Grashoff number (see Goldstein (**5**), p. 607) which is roughly a measure of the ratio between buoyancy and viscous forces. T is a so-called Taylor number which, in measuring the relative dynamical importance of the viscous and Coriolis forces, expresses the influence of rotation.

When $T=0$, equation (22) is quadratic in n and the solution can be easily written down. This case is discussed fully in (**4**). When T is not zero, the equation is cubic in n, which can be shown by rearranging to be

$$n^3+4n^2(s^2+k^2)+n[4(s^2+k^2)^2-(4Gk^2-T)/(s^2+k^2)]-8Gk^2=0. \quad (22')$$

Unstable case, $\beta>0$. When $\beta>0$, G is positive and equation (22') always has one real and positive root. It is this root in which we are interested because it corresponds to instability. It can easily be shown that there is

one mode for which n is greater than that for any other. This is the mode of maximum instability which would be expected to assert itself initially. It will be characterized by the symbols n_m and k_m.

The calculation of n as a function of k was effected for several assigned values of G and T, for the case $s = 1$. On general physical grounds one can see that for given G, T and k, n must decrease with increasing s because the more nodes there are in the velocity field the less effective would the buoyancy forces be, and the more powerful would viscous forces be. One can also see this mathematically by writing down the positive root of equation (22′), namely,

$$n = -k^2(1+s^2/k^2)^2+$$
$$+[k^4(1+s^2/k^2)^4+4G(1+s^2/k^2)-Tn(s^2+k^2)/(n+2s^2+2k^2)]^{\frac{1}{4}}. \quad (25)$$

n is positive only by virtue of G being positive. Any increase in s produces a smaller relative change in $4G(1+s^2/k^2)$ than in the other terms so that the mode of maximum instability would have $s = 1$.

Some of the results of the computations are presented graphically in Figs. 1, 2, and 3.

In Fig. 1 curves of n against k for $G = 1$ and various values of T are given. An inspection of this figure reveals three features, namely, that for a given k, n decreases with increasing T, that n_m decreases with increasing T, and k_m increases with increasing T. Thus, for given G, the effect of rotation (measured by the value of T) is to increase the time taken for the system to depart from equilibrium and to decrease the wavelength of the mode of motion by which the departure occurs.

In Fig. 2, $\log k_m$ is plotted as a function of $\log T$ for several values of G. The principal features to be noted are the following. When T is small, k_m is nearly independent of T and depends only on G. However, when T is large, G is determined entirely by the value of T.

Fig. 3 illustrates the dependence of n_m on T for several values of G.

The asymptotic behaviour of n_m and k_m when T is very large or very small can be deduced from equation (22′). When $T = 0$ we have the case discussed in (4) (see section 7, equations (55) and (56)). T can be ignored even if $T \neq 0$ if

$$4Gk^2 \gg T, \quad (26)$$

and therefore, the results of (4) that

$$k_m = G^{\frac{1}{4}}, \qquad n_m = 2G^{\frac{1}{4}} \quad (G \to \infty) \quad (27)$$

would still be fairly accurate provided that $4G^{\frac{1}{4}} \gg T$ (see (26) above), and

$$k_m = 1, \qquad n_m = G/2 \quad (G \to 0) \quad (28)$$

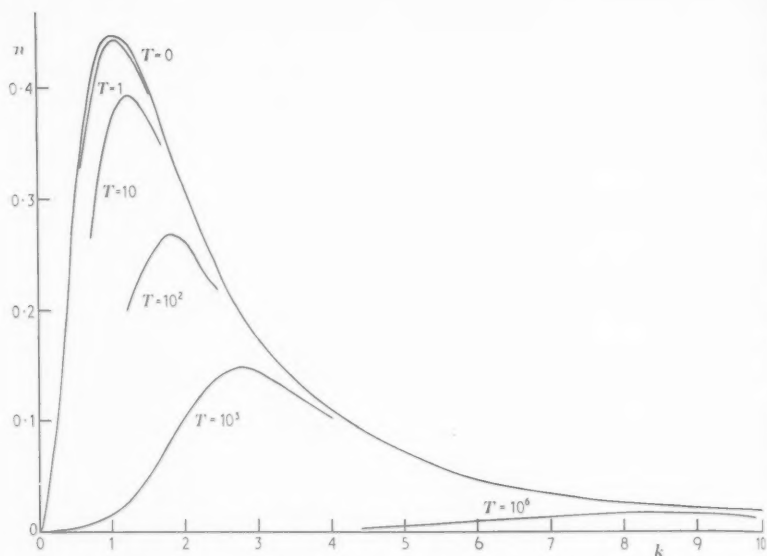provided that $4G \gg T$.

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FIG. 1. Illustrating the inhibiting influence of rotation in the unstable case of a continuously stratified fluid. The growth rate n (measured in the unit $(\nu\pi^2/2d^2)$ sec.$^{-1}$), is plotted as a function of k (measured in the unit (π/d) cm.$^{-1}$), for $G = 1$ and for several values of the Taylor number T.
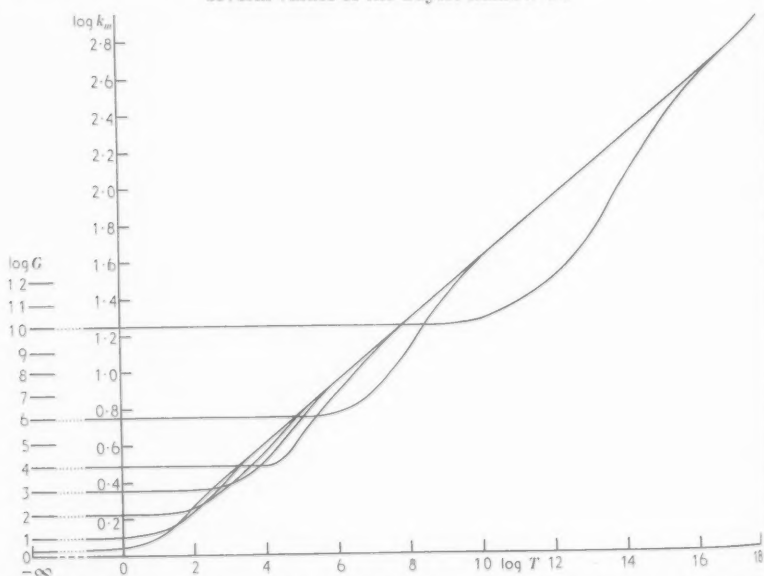

FIG. 2. Continuously stratified fluid in unstable equilibrium $(G > 0)$. The wave number k_m of the mode of maximum instability as a function of G and T.

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The results illustrated in Figs. 2 and 3 show that if G is finite, $k_m \to \infty$ and $n_m \to 0$ as $T \to \infty$. Hence, if

$$4Gk^2 \ll T \quad \text{and} \quad k^2 \gg 1, \tag{29}$$

equation (22′) simplifies to the equation
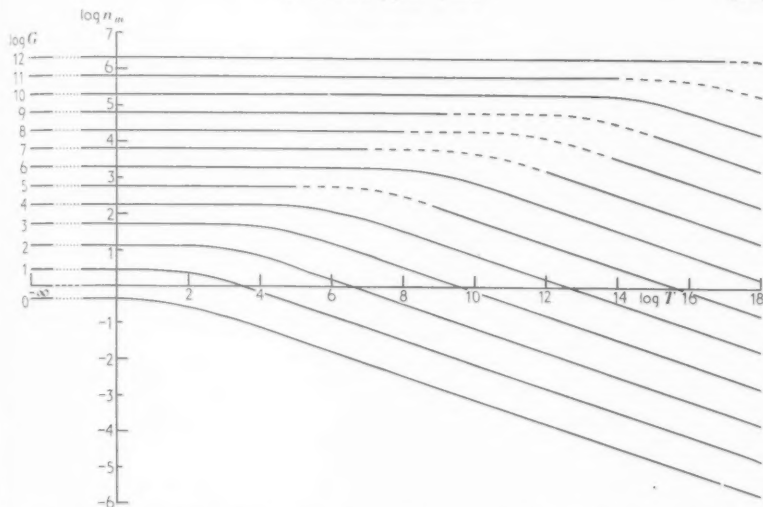
$$n = 8Gk^4/(4k^6 + T) \tag{30}$$

Fig. 3. *Continuously stratified fluid in unstable equilibrium* $(G > 0)$. The growth rate n_m of the mode of maximum instability as a function of G and T. The dashed portions represent interpolation in regions for which no value of n_m was calculated.

to a good approximation. On differentiating this equation with respect to k and setting $(dn/dk) = 0$, it follows that

$$k_m = (T/2)^{\frac{1}{6}} \tag{31}$$

and

$$n_m = (8/3)(4T)^{-\frac{1}{3}}G \quad \text{when} \quad T \to \infty. \tag{32}$$

Equations (31) and (32) are only accurate when (29) are satisfied and writing $k = k_m$ in these inequalities, the conditions for equations (31) and (32) to contain negligible errors are

$$(2T^2)^{\frac{1}{6}} \gg 4G \quad \text{and} \quad (T/2)^{\frac{1}{6}} \gg 1. \tag{33}$$

In order to get a better idea of the implications of the foregoing results, we shall revert to ordinary units. According to (33) rotation seriously inhibits instability if

$$\Omega \gg 0.1892(g\beta)^{\frac{3}{4}}(d^2/\nu)^{\frac{1}{4}} \quad \text{and} \quad \Omega \gg 3.494\nu/d^2, \tag{34}$$

which implies the condition that

$$0.0358(g\beta)^{\frac{3}{2}}(d/\Omega)^2 \ll \nu \ll 0.2862d^2\Omega. \tag{35}$$

When (35) is satisfied then equations (31) and (32) state that

$$n_m = 0\cdot1153g\beta(d^2/\Omega^2\nu)^{\frac{1}{3}} \tag{36}$$

and

$$k_m = 2\cdot071(\nu/\Omega d^2)^{\frac{1}{3}}d^{-1}. \tag{37}$$

When ν is either so small or so large that

$$\nu \leqslant 0\cdot0358(g\beta)^{\frac{3}{2}}(d/\Omega)^2 \quad \text{or} \quad \nu \geqslant 0\cdot2862d^2\Omega, \tag{38}$$

the influence of Coriolis forces can be ignored and the equations (55) and (56) of (4) may be used without fear of serious error. When neither (35) nor (38) is satisfied, n_m and k_m must be estimated from the data given in Figs. 2 and 3.

The inequalities (35) state no more than the condition that rotation effects are significant only when Coriolis forces exceed in strength the viscous forces. Viscous forces would always be predominant if the horizontal shearing stresses, which are proportional to ν/d^2, exceed the Coriolis forces which are proportional to Ω. This is the explanation of the upper limit to ν imposed by (35).

Coriolis forces do not act on motion parallel to the axis of rotation, that is, in the present case, on the vertical flow. Thus, the result that $k_m = \alpha$ when $\nu = 0$, independent of the value of Ω, is not surprising, because $k_m = \infty$ implies no horizontal motion. However, the introduction of a very slight amount of viscosity, so that k_m is still very large, would produce enormous viscous stresses and hence instability would be opposed mainly by viscosity. Only when ν takes the moderate values limited by (35) would k_m be not so large that viscous forces predominate, and only then would the effects of rotation make themselves felt.

Stable case, $\beta < 0$. When β is negative, G is also negative. Let us write $G = -G'$ so that $G' > 0$. Equation (22′) becomes

$$n^3 + 4n^2(1+k^2) + n[4(1+k^2)^2 + (4G'k^2 + T)/(1+k^2)] + 8G'k^2 = 0, \tag{39}$$

when $s = 1$. As k is necessarily real, and T is always positive, n must now be either real and negative, or complex with a negative real part, because all the coefficients in the last equation are now positive.

We know from a previous investigation ((4), section 7) that when $T = 0$, n is real or complex according as

$$4G'k^2/(1+k^2)^3 \lessgtr 1 \tag{40}$$

so that the equilibrium is restored by periodic or aperiodic motion according as $k_1 < k < k_2$ or $k \leqslant k_1$ and $k \geqslant k_2$ where k_1 and k_2 are the two real roots of the equation

$$4G'k^2 = (1+k^2)^3. \tag{41}$$

When

$$G' < 27/16 = G'_{\text{crit}} \text{ (say)} \tag{42}$$

equation (41) has no real roots and hence all modes would be aperiodically damped.

A complete discussion of the influence of T on the foregoing results, and on the properties of the motion, would involve solving equation (39) for n as a function of k for many values of G and T. This has not yet been done. We can, however, learn something about the influence of T by discussing the nature of the roots of equation (39). For a cubic equation of the form

$$n^3+bn^2+cn+d = 0 \tag{43}$$

with real coefficients, there are three real, unequal roots or one real and two complex (conjugate) roots according as the quantity

$$X = (d-bc/3+2b^3/27)^2+(4/27)(c-b^2/3)^3 \lessgtr 0. \tag{44}$$

On substituting for the coefficients from equation (39) we find that

$$X = \frac{1}{16(1+k^2)^3}T^3+\left\{\frac{1}{2}+\frac{3G'k^2}{4(1+k^2)^3}\right\}T^2+ \\ +\left\{(1+k^2)^3-5G'k^2+\frac{3G'^2k^4}{(1+k^2)^3}\right\}T+G'^2k^4\left\{\frac{4G'k^2}{(1+k^2)^3}-1\right\}, \tag{45}$$

or, equivalently,

$$X = \frac{4k^6}{(1+k^2)^3}G'^3+k^4\left\{\frac{3T}{(1+k^2)^3}-1\right\}G'^2+ \\ +Tk^2\left\{\frac{3T}{4(1+k^2)^3}-5\right\}G'+\frac{T}{16}\left\{\frac{T^2}{(1+k^2)^3}+8T+16(1+k^2)^3\right\}. \tag{46}$$

Let us study the effect on X of varying T when $k = k_1$, and $k = k_2$, and restrict attention to values of $G' \gg 1$ so that to a good approximation

$$k_1^2 = (4G')^{-1} \quad \text{and} \quad k_2^2 = (4G')^{\frac{1}{4}} \tag{47}$$

(see equation (41)). On substituting for k in equation (45) we find that

$$X(k_1, T) = (T/16)[T^2+11T-1] \tag{48}$$

and

$$X(k_2, T) = (T/128G'^{\frac{1}{4}})[T^2+88TG'^{\frac{3}{4}}-64G'^3]. \tag{49}$$

Because when $X > 0$ we have damped oscillations and when $X < 0$ the equilibrium is restored aperiodically, the lower limit k_1 to the oscillatory range when $T = 0$ is in the aperiodic range when T is small but not zero. However, when

$$2T = 5\sqrt{5}-11, \tag{50}$$

according to equation (48), $X(k_1, T)$ vanishes and k_1 is again the lower limit of the oscillatory range. When T exceeds the value given by the last equation, k_1 is within the oscillatory range.

$X(k_2, T)$ is also negative when T is small so that k_2 moves initially out of the oscillatory range, but according to equation (49), when

$$T = (20\sqrt{5}-44)G'^{\frac{3}{4}} \tag{51}$$

$X(k_2, T)$ is again zero, and when T exceeds the above value k_2 is to be found in the oscillatory range.

Unfortunately, this is about as far as we can take the discussion of the stable case in the absence of detailed numerical data on the dependence of n on k.

4. The equilibrium of two superposed fluids of great depth

The density configuration is

$$\rho_0(z) = \begin{cases} \rho_1, & 0 > z > -\infty \\ \rho_2, & 0 < z < \infty \end{cases} \tag{52}$$

where ρ_1 and ρ_2 are constants. The coefficients of viscosity μ_1 and μ_2, where the subscripts 1 and 2 refer to the lower and upper fluids respectively, will be assumed to be constants.

Setting $\theta = 0$ and treating ρ_0 and μ_0 as constants, after eliminating ζ between the equations (4) and (5) we obtain the differential equation for w

$$(D^2 - k^2 - n\rho_0/\mu_0)^2(D^2 - k^2)w + (2\Omega\rho_0/\mu_0)^2 D^2 w = 0, \tag{53}$$

which has solutions

$$w(z) = \sum_i A_i e^{\alpha_i z}, \tag{54}$$

where α_i are the roots of the equation

$$(\alpha^2 - q^2)^2\{\alpha^2[1 + 4\Omega^2/(n + \nu_0(k^2 - \alpha^2))^2] - k^2\} = 0 \tag{55}$$

in which we have written

$$q^2 = k^2 + n/\nu_0 \tag{56}$$

and ν_0 is the coefficient of kinematical viscosity, μ_0/ρ_0. An exact treatment would require the evaluation of the α_i's and the determination of the A_i's through an application of the boundary conditions listed in section 2. We shall follow an approximate procedure here.

Let us write

$$\begin{aligned} w_1 &= A_1 e^{-pz} + B_1 e^{+pz} \quad \text{for the lower fluid} \\ w_2 &= A_2 e^{-pz} + B_2 e^{+pz} \quad \text{for the upper fluid} \end{aligned} \tag{57}$$

and

where, apart from requiring re p to be positive, we shall defer specifying its value until later.

The boundary conditions, according to those listed in section 2, are

$$w_1(-\infty) = w_2(+\infty) = 0, \tag{58}$$

$$\zeta_1(-\infty) = \zeta_2(+\infty) = 0, \tag{59}$$

$$w_1(0) = w_2(0), \tag{60}$$

$$Dw_1(0) = Dw_2(0), \tag{61}$$

$$\zeta_1(0) = \zeta_2(0), \tag{62}$$

$$\mu_1(D^2 + k^2)w_1(0) = \mu_2(D^2 + k^2)w_2(0) \tag{63}$$

and δp is continuous at the physical interface.   (64)

Applying the boundary conditions (58) and (60), we find that

$$A_1 = B_2 = 0 \quad \text{and} \quad A_2 = B_1 = A \text{ (say)},$$

so that $$w_1 = A e^{pz} \quad \text{and} \quad w_2 = A e^{-pz}.$$ (65)

If $$\zeta_1 = C_1 e^{pz} \quad \text{and} \quad \zeta_2 = C_2 e^{-pz},$$ (66)

so that (59) is automatically satisfied, substituting for w and ζ in equation (5) leads to

$$C_1 = 2\Omega p A/(n - \nu_1(p^2 - k^2)) \quad \text{and} \quad C_2 = -2\Omega p A/(n - \nu_2(p^2 - k^2)).$$ (67)

It is easily seen that the viscous boundary conditions (61)–(63) cannot be satisfied by the trial functions for w and ζ that we have taken here. However, the condition (64) will be automatically satisfied later if we allow for the discontinuity in ρ_0 at $z = 0$.

We now substitute for w and ζ in the expressions for $I_{1\ldots5}$ (see equations (7)–(11), section 2). Because $w(0)$ and $\rho_0(0)$ are not defined, we must subdivide the range of integration into the ranges $\infty > z > \epsilon, \epsilon > z > -\epsilon$, and $-\epsilon > z > -\infty$, and then proceed to the limit $\epsilon = 0$.

$$I_1 = \int_{-\epsilon}^{\epsilon} \rho_0(z) \left\{ w^2 + \frac{1}{k^2}(Dw)^2 \right\} dz + \frac{A^2}{k^2}(k^2 + p^2)\left\{ \rho_2 \int_{\epsilon}^{\infty} e^{-2pz} dz + \rho_1 \int_{-\infty}^{-\epsilon} e^{2pz} dz \right\}.$$

When we proceed to the limit, because $\rho_0(z)$ is finite and there is no discontinuity in w on $z = 0$, the first contribution to I_1 vanishes, and, on performing the elementary integrations required on the remaining terms we find that $$I_1 = A^2(k^2 + p^2)(\rho_1 + \rho_2)/(2pk^2).$$ (68)

The second integral

$$I_2 = \int_{-\epsilon}^{\epsilon} (D\rho_0)w^2 dz = \int_{\rho_1}^{\rho_2} w^2 d\rho_0$$

because ρ_0 is constant except within the region $\epsilon > z > -\epsilon$, where it changes from ρ_2 to ρ_1. The mean value of w^2 on $z = 0$ is A^2 so that

$$I_2 = (\rho_2 - \rho_1)A^2.$$ (69)

The third integral

$$I_3 = \frac{A^2}{k^2}(k^2 + p^2)^2 \left\{ \mu_2 \int_{\epsilon}^{\infty} e^{-2pz} dz + \mu_1 \int_{-\infty}^{-\epsilon} e^{2pz} dz \right\} +$$
$$+ \int_{-\epsilon}^{\epsilon} \mu_0(z) \left\{ k^2 w + 2(Dw)^2 + \frac{1}{k^2}(D^2w)^2 \right\} dz + \int_{-\epsilon}^{\epsilon} (D^2\mu_0)w^2 dz.$$

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The second contribution vanishes in the limit $\epsilon = 0$ because μ_0 is finite and w and Dw must be continuous. The last term on the right-hand side of the last equation vanishes by virtue of the fact that $[D\mu_0]_{\pm\epsilon} = 0$. Hence, it follows that

$$I_3 = [A^2(k^2+p^2)^2/2k^2p](\mu_1+\mu_2). \tag{70}$$

The fourth integral

$$I_4 = \frac{1}{k^2}\int_{-\epsilon}^{\epsilon} \rho_0(z)\zeta^2\,dz + \frac{1}{k^2}\left\{C_2^2\rho_2\int_{\epsilon}^{\infty} e^{-2pz}\,dz + C_1^2\rho_1\int_{-\infty}^{-\epsilon} e^{2pz}\,dz\right\}.$$

Because $\rho_0(z)$ is finite and $\zeta(0)$ must be continuous,

$$I_4 = \frac{\rho_2 C_2^2+\rho_1 C_1^2}{2pk^2} = \frac{2\Omega^2 pA^2}{k^2}\left\{\frac{\rho_2}{[n-\nu_2(p^2-k^2)]^2}+\frac{\rho_1}{[n-\nu_1(p^2-k^2)]^2}\right\}, \tag{71}$$

after substituting for C_1 and C_2 from equation (67).

The last integral

$$I_5 = \int_{-\epsilon}^{\epsilon} \mu_0(z)\left\{\zeta^2+\frac{1}{k^2}(D\zeta)^2\right\}dz + \frac{k^2+p^2}{k^2}\left\{\mu_2 C_2^2\int_{\epsilon}^{\infty} e^{-2pz}\,dz +\mu_1 C_1^2\int_{-\infty}^{-\epsilon} e^{2pz}\,dz\right\},$$

so that when $\epsilon = 0$,

$$I_5 = \frac{2\Omega^2 pA^2(k^2+p^2)}{k^2}\left\{\frac{\mu_2}{[n-\nu_2(p^2-k^2)]^2}+\frac{\mu_1}{[n-\nu_1(p^2-k^2)]^2}\right\} \tag{72}$$

because the first contribution vanishes.

On substituting for the I's in equation (6) we obtain the result

$$n^2\left\{\frac{(p^2+k^2)(\rho_1+\rho_2)}{2pk^2}+\frac{2\Omega^2 p}{k^2}\left[\frac{\rho_2}{[n-\nu_2(p^2-k^2)]^2}+\frac{\rho_1}{[n-\nu_1(p^2-k^2)]^2}\right]\right\}+$$
$$+n\left\{\frac{(k^2+p^2)^2(\mu_1+\mu_2)}{2k^2p}+\frac{2\Omega^2 p(p^2+k^2)}{k^2}\times\right.$$
$$\left.\times\left[\frac{\mu_2}{[n-\nu_2(p^2-k^2)]^2}+\frac{\mu_1}{[n-\nu_1(p^2-k^2)]^2}\right]\right\}-g(\rho_2-\rho_1) = 0, \tag{73}$$

which reduces to equation (17) of (4) when $\Omega = 0$ if we write $p = k$.

Equation (73) cannot be easily discussed without making some simplifications. Let us divide our discussion into two parts, namely, the 'high' and 'low' rotation cases, and concentrate on discovering the characteristics of the mode of maximum instability when $\rho_2 > \rho_1$. Because there is no fixed length scale in the system, we must only require that the Coriolis forces exceed or are less than the other forces operating, that is, that

$$\nu k_m^2 \lessgtr \Omega \quad \text{and} \quad n_m \lessgtr \Omega \tag{74}$$

according as we are dealing with 'high rotation' or 'low rotation'.

Instability in the 'high rotation' case. We know that when $\nu = 0$

$$p = k(1+4\Omega^2/n^2)^{\frac{1}{2}}, \quad (75)$$

a fact that follows immediately from equation (55). If we use this as trial function, and in order to simplify equation (73) set $\nu_1 = \nu_2 = \nu$, then

$$n^2\left\{1 - \frac{2\Omega^2}{n^2}\left[1 - \frac{1}{[1+(\nu k^2/n)/(1+n^2/4\Omega^2)]^2}\right]\right\} +$$
$$+ 2\nu n k^2\left\{\frac{n^2+2\Omega^2}{n^2+4\Omega^2}\right\}\left\{1 - \frac{2\Omega^2}{n^2+4\Omega^2}\left[1 - \frac{1}{[1+(\nu k^2/n)/(1+n^2/4\Omega^2)]^2}\right]\right\} -$$
$$- gk\alpha(1+4\Omega^2/n^2)^{-\frac{1}{2}} = 0, \quad (76)$$

where we have written

$$\alpha = (\rho_2-\rho_1)/(\rho_2+\rho_1). \quad (77)$$

Let us make the following approximations. Assume that

$$1 \gg (1+\nu k^2/n)^{-2} \quad (78)$$

so that we can ignore the right-hand side of this inequality compared with unity, and that $n^2 \ll \Omega^2$, then to the first approximation equation (76) leads to the very simple result that

$$n = gk\alpha/\Omega - k^2\nu. \quad (79)$$

It follows immediately from the last equation that there is a mode of maximum instability for which

$$k_m = (g\alpha/2\nu\Omega) \quad \text{and} \quad n_m = (g^2\alpha^2/4\nu\Omega^2). \quad (80)$$

For these expressions to be reasonably accurate, according to (74) we require that the inequality

$$(g^2\alpha^2/4\nu\Omega^3) \ll 1 \quad (81)$$

be satisfied. The assumption (78) is likely to be a fairly good one; on substituting for n and k the values n_m and k_m we find that the right-hand side takes the value 0·25.

Instability in the 'low rotation' case. This is the case in which viscosity resists the motion much more than rotation. The solution in the limiting case $\Omega = 0$ is known, namely

$$n_m = (g^2\alpha^2/8\nu)^{\frac{1}{3}} \quad (82)$$

and

$$k_m = (g\alpha/8\nu^2)^{\frac{1}{3}} \quad (83)$$

(see (4), equations (21) and (22)). The criteria (74) tell us that the values of k_m and n_m predicted by the last two equations should not be too much in error even when $\Omega \neq 0$, provided that

$$(g^2\alpha^2/4\nu\Omega^3) \gg 1. \quad (84)$$

A second approximation may be obtained by writing $p = k$ in equation

(73) and retaining only those terms not less than first order in (Ω^2/n^2). Thus

$$n^2 + 2\nu n k^2 - g\alpha k(1 - 2\Omega^2/n^2) = 0. \qquad (85)$$

On differentiating the last equation with respect to k and setting $(dn/dk) = 0$ we find that

$$n_m k_m = g\alpha(1 - 2\Omega^2/n_m^2)/4\nu \qquad (86)$$

so that we have, on combining the last two equations,

$$n_m = \left(\frac{g^2\alpha^2}{8\nu}\right)^{\frac{1}{3}}\left\{1 - \frac{16}{3}\left(\frac{\Omega^3\nu}{g^2\alpha^2}\right)^{\frac{2}{3}}\right\} \qquad (87)$$

and

$$k_m = \left(\frac{g\alpha}{8\nu^2}\right)^{\frac{1}{3}}\left\{1 - \frac{8}{3}\left(\frac{\Omega^3\nu}{g^2\alpha^2}\right)^{\frac{2}{3}}\right\}. \qquad (88)$$

Thus, the effect of slow rotation is to increase the wavelength, $(2\pi/k_m)$, of the mode of maximum instability, and to decrease its growth rate.

It is convenient to express the results of this section in the following form. Let

$$n_m = \Omega f(Y) \quad \text{and} \quad k_m = (\Omega^2/g\alpha)h(Y), \qquad (89)$$

where Y is the pure number

$$Y \equiv (g^2\alpha^2/\nu\Omega^3). \qquad (90)$$

Then equations (87), (88), and (80) become

$$n_m = \tfrac{1}{2}\Omega Y^{\frac{1}{3}}(1 - \tfrac{16}{3}Y^{-\frac{2}{3}}) \qquad (91)$$

and

$$k_m = \tfrac{1}{2}(\Omega^2/g\alpha)Y^{\frac{2}{3}}(1 - \tfrac{8}{3}Y^{-\frac{2}{3}}), \qquad (92)$$

when $Y \gg 4$, and

$$n_m = \tfrac{1}{4}\Omega Y \qquad (93)$$

and

$$k_m = \tfrac{1}{2}(\Omega^2/g\alpha)Y, \qquad (94)$$

when $Y \ll 4$.

In conclusion, I must record my thanks to Professor S. Chandrasekhar, with whom it was my privilege to study for one year at the Yerkes Observatory, for the benefit of many stimulating discussions.

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THE EFFECT OF DIHEDRAL ON THE LIFT AND DRAG COEFFICIENTS OF AIRFOILS IN SUPERSONIC FLOW

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SUMMARY

The pressure distribution on the surface of a dihedral wing in supersonic flow is investigated for varying angles of sweepback and Mach numbers. A solution is obtained by the linearized theory on the assumption that the angles of incidence and yaw are small. The results are then used to find the effects produced by dihedral on the lift and drag coefficients of the wing. In particular, the results for a delta wing are exhibited graphically.

If the angle of dihedral be α, and the angle of incidence be measured in a vertical plane, then ignoring the effects of each half of the wing on the other, it is clear that a dihedral α causes the lift coefficient to be reduced by a factor $\cos^2\alpha$. It is found that for a delta wing this simple rule is a useful approximation, for example, when $\alpha = 30°$, the maximum error is less than 8 per cent. The greatest error occurs at Mach numbers such that the wing leading edges lie near the surface of the Mach cone.

Notation

A, B	constants defined by equations (50) and (57).
a, b, c	parameters introduced in section 6 (see equation 52).
a_1, b_1, c_1	parameters introduced in section 6.
C_L, C_D	lift and drag coefficients.
C'_L, C'_D	modified lift and drag coefficients introduced in section 5.
$E(q)$	complete elliptic integral of the second kind of modulus q.
f, g	functions defining the upper and lower surfaces of the wing in section 5.
i	$= (-1)^{\frac{1}{2}}$.
K	parameter of the transformation (41).
l	parameter defined by equation (38).
M	Mach number.
p	pressure.
p_2	pressure inside the Mach cone.
p_1	pressure inside the Mach wedge but outside the Mach cone (see section 3).

p_3, p_4	dimensionless pressures.
q	$= \dfrac{1-l^2}{1+l^2}.$
r, θ, z	polar coordinates in the physical plane.
R	$= \dfrac{\beta r}{z}.$
t	parameter of small order used in section 7.
U	velocity of incident stream.
V_r, V_θ, V_z	velocity components in the polar coordinate plane.
$z_1 = x_1 + iy_1$	coordinates of the first transformed plane.
$z_2 = x_2 + iy_2$	coordinates of the second transformed plane.
α	angle of dihedral.
β	$= (M^2 - 1)^{\frac{1}{2}}.$
Γ	circle of integration (appendix).
γ_1, γ_2	constants defined by equation (17).
γ, δ	constants defined by equations (42) and (44).
ϵ	angle of incidence.
$\zeta = \xi + i\eta$	coordinates of the third transformed plane.
λ	angle of sweepback.
μ	variable of integration.
ν	variable of integration.
ρ	radial polar coordinate of the first transformed plane.
ρ_0	density at infinity.
$U\phi$	velocity potential.
ψ	parameter defined by equation (14).
σ	radius of the circle Γ (appendix).
τ	$= \dfrac{\pi(\pi - 2\psi)}{2(\pi - 2\alpha)}.$

1. Introduction

THE linearized flow theory has been used to great advantage for solving problems concerning two-dimensional thin airfoils in supersonic flow. In particular, the problem of supersonic flow over a flat delta wing at incidence has been solved by Puckett (1), and later by others using the methods of cone-field theory. Since then, the methods of conformal transformations have also been applied by Stewart (2) for the case of a delta wing lying entirely inside the Mach cone of the vertex and by Roper (3, 4) who has used a similar method for obtaining the solution in those cases which arise when the wing cuts the Mach cone.

In this paper the solution is extended to include the case of wings possessing a dihedral angle. Goldstein and Ward (5), in a paper which

considerably unifies the previous work on the subject, mention this problem for the wing lying entirely inside the Mach cone of the vertex. However, the method used here, although employing cone-field flow and the useful theory of functions of a complex variable, differs from that suggested by Goldstein and Ward. Nevertheless, by using the Busemann reduction of the wave equation (6), the problem is again reduced to that of solving Laplace's equation under convenient boundary conditions.

The problem is developed for both swept-back and unswept wings, of which those of wedge-shaped cross-section are considered the most fully. Computations of the lift coefficient are made for the case when the wedge angle is zero. The solution for an airfoil of arbitrary cross-section may then be obtained in the usual manner, by considering the wing as a number of arbitrary variations in slope, to each of which the present theory may be applied. The wings are assumed to be of infinite span. For swept-back wings this is not necessary provided that the trailing edges are such that all Mach cones leaving them do not further cut the wing. (We shall in fact assume the swept-back wing to be delta-shaped for the purposes of calculating lift and drag coefficients. Clearly any shape may be chosen within the limitations of the theory, and for this reason we shall introduce so-called modified lift and drag coefficients.) The two cases of the wing lying entirely inside and partly outside the vertex Mach cone are both considered.

2. Formulation: transformation of the equations

Let the velocity, pressure, and density of the undisturbed flow be U, p_0, and ρ_0 respectively. With the origin at the vertex of the leading edge, take cylindrical polar coordinates (r, θ, z) where the z-axis is chosen to be in the direction of the incident stream, so that the wings lie approximately in the planes $\theta = \alpha$, $\theta = \pi - \alpha$. We then define α to be the angle of dihedral.

Neglecting viscosity, heat conduction, and radiation, we assume that the variation of the flow velocities is small compared with U, and hence we may write the perturbation velocity components as first order quantities (V_r, V_θ, V_z). If the flow is also irrotational, there exists a velocity potential, which we may reduce to linear dimensions by expressing it in the form $U\phi$ so that

$$V_r = U\frac{\partial\phi}{\partial r}, \qquad V_\theta = U\frac{1}{r}\frac{\partial\phi}{\partial\theta}, \qquad V_z = U\frac{\partial\phi}{\partial z}. \tag{1}$$

This potential may then be shown to satisfy the well-known Prandtl–Glauert equation

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \beta^2\frac{\partial^2\phi}{\partial z^2} \tag{2}$$

where $\beta^2 = M^2 - 1$, and M is the Mach number of the incident flow.

Bernoulli's equation may also be linearized to obtain the pressure p at any point in terms of $\dot{\phi}$, viz.

$$p = p_0 - \rho_0 U^2 \frac{\partial \phi}{\partial z}. \tag{3}$$

This indicates that to the first order the pressure, too, is a solution of the wave equation (2). It will now be convenient to consider the problem in terms of the pressure p, and solve the wave equation with respect to the boundary conditions imposed on p and its derivatives on the wing and the Mach cone.

The problem, as stated, involves no fundamental length, and so we may use the concept of conical flow. Hence the dimensionless quantity $p/\rho_0 U^2$ is a function of the independent variables r/z and θ only. This renders it possible for the following transformation into the polar coordinate plane (ρ, θ) due to Busemann (6), to be made

$$\frac{\beta r}{z} = R = \frac{2\rho}{1+\rho^2}. \tag{4}$$

The transformation from the (R, θ) to the (ρ, θ) plane leaves invariant the unit circle and all angular distances, whilst the whole transformation (4) transforms equation (2) for the pressure p into Laplace's equation in the polar coordinates (ρ, θ),

$$\frac{\partial^2 p}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial p}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 p}{\partial \theta^2} = 0. \tag{5}$$

The Mach cone with vertex at the origin now becomes the unit circle in the (ρ, θ) plane, and we require to solve Laplace's equation inside the circle $\rho = 1$.

If the relations (1) are substituted in equation (2) and taken together with the irrotationality conditions, we obtain a set of equations which are satisfied by the perturbation velocities,

$$\frac{\partial}{\partial r}V_r + \frac{1}{r}V_r + \frac{1}{r}\frac{\partial}{\partial \theta}V_\theta = \beta^2 \frac{\partial}{\partial z}V_z, \tag{6}$$

$$\frac{\partial}{\partial r}V_z = \frac{\partial}{\partial z}V_r, \tag{7}$$

$$r\frac{\partial}{\partial z}V_\theta = \frac{\partial}{\partial \theta}V_z. \tag{8}$$

We transform these equations into the coordinates (ρ, θ) by equation (4) and use the fact that by the cone-field theory, V_r, V_θ, and V_z are functions

of ρ and θ alone. Hence

$$\frac{\partial}{\partial\rho}V_z = -\frac{2\rho}{\beta(1+\rho^2)}\frac{\partial}{\partial\rho}V_r = \frac{2}{\beta(1-\rho^2)}V_r + \frac{2}{\beta(1-\rho^2)}\frac{\partial}{\partial\theta}V_\theta, \tag{9}$$

$$\frac{\partial}{\partial\theta}V_z = -\frac{2\rho^2}{\beta(1-\rho^2)}\frac{\partial}{\partial\rho}V_\theta, \tag{10}$$

which in view of equation (3) may be rewritten in the form

$$\frac{\partial p}{\partial\rho} = -\frac{2\rho_0\,U}{\beta(1-\rho^2)}V_r - \frac{2\rho_0\,U}{\beta(1-\rho^2)}\frac{\partial}{\partial\theta}V_\theta, \tag{11}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial\theta} = \frac{2\rho_0\,U\rho}{\beta(1-\rho^2)}\frac{\partial}{\partial\rho}V_\theta. \tag{12}$$

Because of the differing boundary conditions, it will be necessary at this stage to consider the problem in two parts. The first is that for which the wing leading edge lies outside the Mach cone of the origin; the second is that for which the wing lies entirely inside the Mach cone. In both of these, the angle of incidence of a wing is measured from the line of undisturbed flow in the vertical plane through that line, and is taken to be positive when measured in a downward direction. Upon investigating the case of the leading edge outside the Mach cone, it will be seen that the boundary conditions for the lower region do not affect the solution in the upper region, and thus initially we need consider only one face of the wing.

3. The wing with the leading edge outside the Mach cone

In this section we shall investigate the pressure on a flat plate wing which, for the purposes of this paper, is taken to mean a dihedral wing whose halves consist of flat plates meeting in the plane $\theta = \frac{1}{2}\pi$. As has previously been mentioned, the results may then be extended by a linearized theory to include more general wings. The wing is supposed to be semi-infinite in the z-direction, of infinite span and dihedral angle α. We also take the angle of incidence to be a first-order quantity ϵ, and the sweepback to be λ, here defined to be the angle between the leading edge and the plane $z = 0$. Thus $\lambda = 0$ corresponds to the unswept wing, which is also included in this investigation.

The pressure in that region of the Mach wedge from the leading edge, which lies outside the Mach cone from the origin, is denoted by p_1 (see Fig. 1). Then p_1 differs only slightly from p_0 and will be given by the Prandtl–Meyer theory of the first order, viz.

$$p_1 = p_0 - \frac{1}{\beta}\rho_0\,U^2\,\epsilon\cos\alpha\sec\psi, \tag{13}$$

where the angle ψ is given by

$$\sin\psi = \frac{1}{\beta}\tan\lambda. \tag{14}$$

From this expression, it is clear that ψ is always defined, except when the wing lies totally inside the Mach cone. We shall consider this case later.

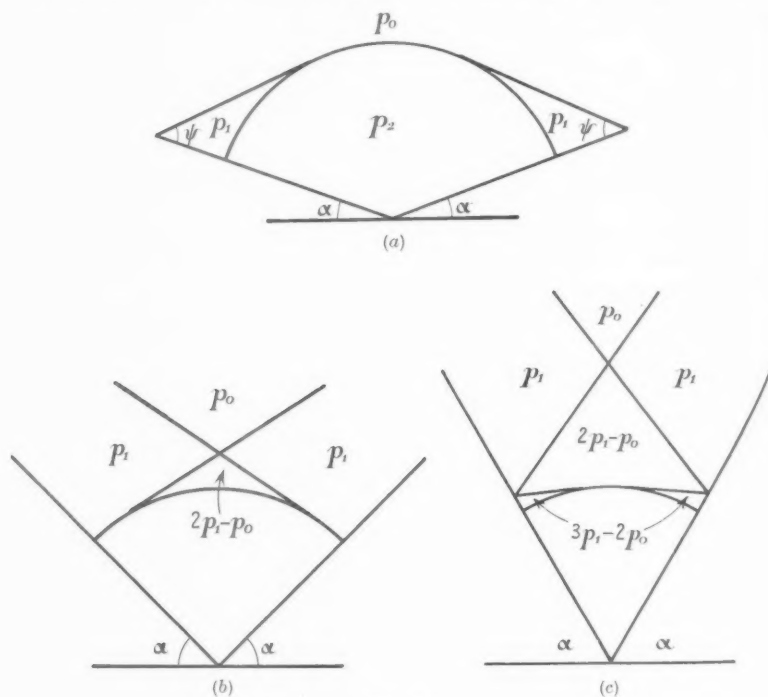To the first order, the Mach wedges from the leading edges may be treated

FIG. 1

as acoustic waves obeying the laws of reflection. In the physical plane, these wedges have the same angle as the angle of the Mach cone, and hence either before or after reflection at the wing surfaces, they meet this cone as tangents. In the (R, θ) plane they will be represented as lines which again ultimately meet the unit circle as tangents (see Fig. 1). (In fact, the wave equation (2) in the cone-field coordinates (R, θ) is hyperbolic outside the unit circle and its characteristics are all tangents to the unit circle.) These lines, along which the discontinuities in the pressure are propagated make, before reflection, angles with the lines $\theta = \alpha$, $\theta = \pi - \alpha$ which may easily be shown to be equal to the ψ defined in equation (14). Owing to the

acoustic nature of the discontinuities, they may be continued across intersections of the Mach wedges. Hence if we write p_2 as the pressure inside the unit circle, the boundary values of p_2 on $\rho = 1$ will depend on the number of reflections of these waves. Three examples of these, shown in Fig. 1, are

(a) for $\alpha < \psi$

$$p_2 = p_1 \qquad \alpha < \theta < \tfrac{1}{2}\pi + \alpha - \psi$$
$$p_2 = p_0 \qquad \tfrac{1}{2}\pi + \alpha - \psi < \theta < \tfrac{1}{2}\pi - \alpha + \psi$$
$$p_2 = p_1 \qquad \tfrac{1}{2}\pi - \alpha + \psi < \theta < \pi - \alpha$$

(b) for $\psi < \alpha < \tfrac{1}{4}\pi + \tfrac{1}{2}\psi$

$$p_2 = p_1 \qquad \alpha < \theta < \tfrac{1}{2}\pi - \alpha + \psi$$
$$p_2 = 2p_1 - p_0 \qquad \tfrac{1}{2}\pi - \alpha + \psi < \theta < \tfrac{1}{2}\pi + \alpha - \psi$$
$$p_2 = p_1 \qquad \tfrac{1}{2}\pi + \alpha - \psi < \theta < \pi - \alpha$$

(c) for $\tfrac{1}{4}\pi + \tfrac{1}{2}\psi < \alpha < \tfrac{1}{3}\pi + \tfrac{1}{3}\psi$

$$p_2 = 3p_1 - 2p_0 \qquad \alpha < \theta < 3\alpha - \tfrac{1}{2}\pi - \psi$$
$$p_2 = 2p_1 - p_0 \qquad 3\alpha - \tfrac{1}{2}\pi - \psi < \theta < \tfrac{3}{2}\pi - 3\alpha + \psi$$
$$p_2 = 3p_1 - 2p_0 \qquad \tfrac{3}{2}\pi - 3\alpha + \psi < \theta < \pi - \alpha$$

Suppose now that we introduce a dimensionless pressure, by writing

$$p_3 = \frac{p_2 - p_1}{p_1 - p_0}. \tag{15}$$

The boundary conditions on the unit circle in the (ρ, θ) plane, may then be written in a form which includes all the cases that occur, viz. on $\rho = 1$, $\partial p_3 / \partial \theta = 0$ together with the additional condition

$$\lim_{\delta \to 0} \int_{-\delta + \gamma_1}^{\delta + \gamma_1} \frac{\partial p_3}{\partial \theta}\, d\theta = -1, \qquad \lim_{\delta \to 0} \int_{-\delta + \gamma_2}^{\delta + \gamma_2} \frac{\partial p_3}{\partial \theta}\, d\theta = +1, \tag{16}$$

where γ_1, γ_2 are such that

$$\gamma_1 = \tfrac{1}{2}\pi + \alpha - \psi \quad \mathrm{mod}(\pi - 2\alpha),$$
$$\gamma_2 = \tfrac{1}{2}\pi - \alpha + \psi \quad \mathrm{mod}(\pi - 2\alpha). \tag{17}$$

As the incidence is supposed small, and we are assuming cone-field flow, the boundary conditions on the wings are applied on the planes $\theta = \alpha$, $\theta = \pi - \alpha$ which in the (ρ, θ) plane are the lines given by the same coordinates. The boundary condition that there shall be no normal velocity on the wing $\theta = \alpha$ gives $V_\theta = -U\epsilon \cos \alpha$. Substituting this in equation (12), we see that on the wing $\theta = \alpha$, $\partial p_3 / \partial \theta = 0$, with a similar result for the wing $\theta = \pi - \alpha$.

Let (x_1, y_1) be cartesian coordinates corresponding to the plane of the

polar coordinates (ρ, θ); then writing $z_1 = x_1 + iy_1$ we consider the conformal transformation from the z_1-plane to the $\zeta \, (= \xi + i\eta)$ plane, where

$$\zeta = -\frac{1}{2}\left\{z_2 + \frac{1}{z_2}\right\} \tag{18}$$

and

$$z_2 = \{z_1 e^{-i\alpha}\}^{\pi/(\pi-2\alpha)}. \tag{19}$$

These map the region bounded by the unit circle and the wings on to the upper half-plane. That part of the boundary consisting of $\theta = \alpha \, (0 \leqslant \rho \leqslant 1)$ in the z_1-plane, then becomes $\eta = 0$, $-\infty < \xi \leqslant -1$ in the ζ-plane, the unit circle between $\theta = \alpha$ and $\theta = \pi - \alpha$ becomes $\eta = 0$, $-1 \leqslant \xi \leqslant +1$; and finally $\theta = \pi - \alpha \, (0 \leqslant \rho \leqslant 1)$ becomes $\eta = 0$, $1 \leqslant \xi < \infty$. In the ζ-plane those points corresponding to γ_1, γ_2 on the unit circle in the z_1-plane are $\xi = \pm\cos\tau$, $\eta = 0$ where

$$\tau = \tfrac{1}{2}\pi\left[\frac{\pi - 2\psi}{\pi - 2\alpha}\right]. \tag{20}$$

The boundary condition across these points follows from (16);

$$\lim_{\delta \to 0} \int_{-\delta+\cos\tau}^{\delta+\cos\tau} \frac{\partial p_3}{\partial \xi}\, d\xi = \frac{\sin\tau}{|\sin\tau|}, \qquad \lim_{\delta \to 0} \int_{-\delta-\cos\tau}^{\delta-\cos\tau} \frac{\partial p_3}{\partial \xi}\, d\xi = -\frac{\sin\tau}{|\sin\tau|}, \tag{21}$$

whilst the remaining boundary conditions on the real axis in the ζ-plane may be written as $\dfrac{\partial p_3}{\partial \xi} = 0$ for $|\xi| < 1$, $\dfrac{\partial p_3}{\partial \eta} = 0$ for $|\xi| > 1$.

We now define the complex function $W(\zeta)$ by

$$W(\zeta) = \frac{\partial p_3}{\partial \eta} + i\frac{\partial p_3}{\partial \xi}. \tag{22}$$

The boundary conditions expressed in terms of W are that, on the real axis, W is real for $|\xi| < 1$, and wholly imaginary for $|\xi| > 1$. Near $\zeta = \cos\tau$, we have by (21),

$$W \sim -\frac{1}{\pi}\frac{\sin\tau}{|\sin\tau|}\frac{1}{\zeta - \cos\tau}$$

whilst near $\zeta = -\cos\tau$ we have

$$W \sim +\frac{1}{\pi}\frac{\sin\tau}{|\sin\tau|}\frac{1}{\zeta + \cos\tau}.$$

Since p_3 is a solution of Laplace's equation, it follows from (22) that W is a regular function of ζ. Moreover p_3 must be bounded at the centre span, and hence $W = O(\zeta^t)$ as $\zeta \to \infty$, where $t < -1$. A function which satisfies all the above conditions, and is regular and integrable at all points in the upper half-plane except the specified singularities is

$$W = \frac{1}{\pi}\frac{\sin\tau}{(1-\zeta^2)^{\frac{1}{2}}}\left[\frac{1}{\zeta + \cos\tau} - \frac{1}{\zeta - \cos\tau}\right], \tag{23}$$

where the branch of $(1-\zeta^2)^{\frac{1}{2}}$ is taken to be that which is real and positive for $|\zeta| < 1$ on the real axis.

The pressure on the wing can now be calculated, for when $z_1 = \rho e^{i\alpha}$ ζ is real and less than -1; so by (23)

$$\frac{\partial p_3}{\partial \xi} = -\frac{\sin\tau}{\pi(\xi^2-1)^{\frac{1}{2}}}\left[\frac{1}{\xi+\cos\tau} - \frac{1}{\xi-\cos\tau}\right]. \tag{24}$$

Integrating this result, the pressure on the wing is obtained fro

$$p_3 = \pm\frac{1}{\pi}\arccos\left[\frac{\pm\sin^2\tau(\xi^2-1)}{\xi^2-\cos^2\tau} + \frac{1-\xi^2\cos^2\tau}{\xi^2-\cos^2\tau}\right] + \text{constant}.$$

Since the pressure inside the unit circle must be constant when there is no dihedral or sweepback, we should expect p_3 to be constant when $\tau = \frac{1}{2}\pi$. Thus we choose the positive sign inside the bracket. Let the range of inverse cosine lie between 0 and π, then the remaining indeterminate sign and the constant of integration will depend on the number of reflections of the Mach wedges.

(a) For $\alpha < \psi$ (i.e. for $\tau < \frac{1}{2}\pi$) the pressure p_2 on the wing, at the point where it is cut by the Mach cone must be equal to p_1; thus we have $p_3 = 0$ at $\xi = \pm 1$. Hence, there is no constant of integration, and moreover we choose the negative sign since the boundary conditions for p_3 on $\rho = 1$ are all non-positive.

$$p_3 = -\frac{1}{\pi}\arccos\left[\frac{\cos^2\tau-\xi^2\cos 2\tau}{\xi^2-\cos^2\tau}\right]. \tag{25}$$

(b) For $\psi < \alpha < \frac{1}{4}\pi+\frac{1}{2}\psi$, we again require that p_3 shall be zero at $\xi = \pm 1$, but here the boundary conditions for p_3 on $\rho = 1$, are all non-negative and so the positive sign is taken.

(c) For $\frac{1}{4}\pi+\frac{1}{2}\psi < \alpha < \frac{1}{3}\pi+\frac{1}{3}\psi$ there is the condition $p_3 = 2$ at $\xi = \pm 1$; and at $\alpha = \frac{1}{4}\pi+\frac{1}{2}\psi$ the value of p_3 in this case must agree with that in the previous case. Hence

$$p_3 = 2-\frac{1}{\pi}\arccos\left[\frac{\cos^2\tau-\xi^2\cos 2\tau}{\xi^2-\cos^2\tau}\right].$$

Similarly for the other cases that arise.

Reverting to equations (4), (18), and (19), there is a relation between ξ and $\beta r/z$ or R of the form

$$\frac{\beta r}{z} = R = \frac{2\rho}{1+\rho^2}, \qquad \xi = -\frac{1}{2}(\rho^{\pi/(\pi-2\alpha)}+\rho^{-\pi/(\pi-2\alpha)}). \tag{26}$$

This enables the pressure to be calculated at any point of the wing. For values of R in the range $0\cdot0$, $(0\cdot1)$, $1\cdot0$, the non-dimensional pressure p_3 has been plotted in Figs. 2 and 3, in two cases, for suitable angles of

dihedral α. The first case is that for no sweepback, the second is for a sweepback and Mach number combining by (14) to give $\psi = 40°$.

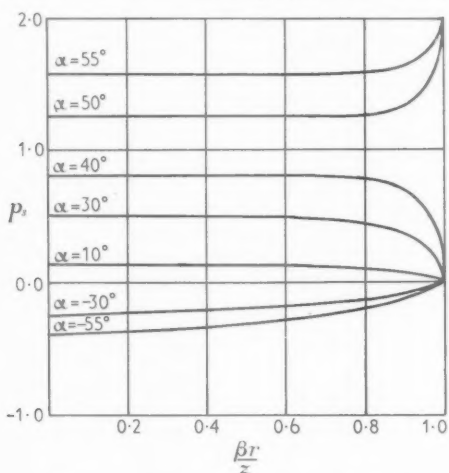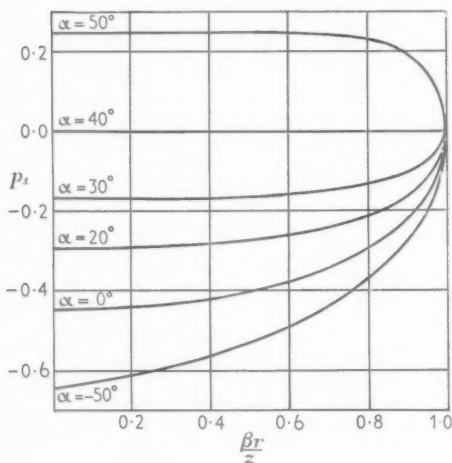

FIG. 2. Pressure distribution for $\psi = 0°$.

FIG. 3. Pressure distribution for $\psi = 40°$.

If we use equation (15) to obtain the pressure in dimensional form we may write equation (25) in the alternative form

$$p_2 - p_0 = \frac{2}{\pi}(p_1 - p_0)\arcsin\left[\frac{\xi\sin\tau}{(\xi^2 - \cos^2\tau)^{\frac{1}{2}}}\right]. \tag{27}$$

For the case of no dihedral, i.e. $\alpha = 0$, we have by (26) $\xi = -z/\beta r$. Thus, substituting for $p_1 - p_0$ from (13) and writing $\frac{1}{2}\pi - \psi$ for τ,

$$p_2 - p_0 = -\frac{2\rho_0 U^2 \epsilon}{\pi \beta \cos \psi} \arcsin \left[\frac{\cos \psi}{\{1 - (r^2/z^2)\tan^2 \lambda\}^{\frac{1}{2}}} \right],$$

a result which agrees with the solutions previously obtained by Ward (7) and Miss Roper (3). A case of further interest is that obtained when the leading edge approaches the Mach cone from the outside. This is given as ψ approaches $\frac{1}{2}\pi$ or the limit as τ tends to zero. From equation (13) it is clear that $p_1 - p_0$ has no finite limit as $\tau \to 0$. However, if we first substitute for $p_1 - p_0$ in equation (27), the limit of $p_2 - p_0$ is finite and given by

$$p_2 - p_0 = -\frac{2\rho_0 U^2 \epsilon \cos \alpha}{\beta(\pi - 2\alpha)} \left[\frac{1 + \rho^{2\pi/(\pi - 2\alpha)}}{1 - \rho^{2\pi/(\pi - 2\alpha)}} \right]. \tag{28}$$

4. Lift and drag coefficients of a delta wing

Let the delta wing consist of a flat plate bent to a dihedral α defined as in section 2. If the sweepback and incidence are λ and ϵ respectively, we shall consider the case when the leading edge of the wing lies outside the vertex Mach cone; the other possibility will be considered in section 7. The lift on the upper surface of the wing may be considered as being due to two pressure distributions. The first is the constant pressure p_1 acting over the entire wing, the second is the pressure $p_2 - p_1$ present on that part of the wing inside the Mach cone.

This last term will give the normal force on the upper surface, as

$$-\iint (p_2 - p_1) \, dr dz,$$

where (r, θ, z) are the cylindrical polar coordinates of the physical plane, introduced in section 2, and the integral is taken over that part of the wing inside the Mach cone. If the trailing edge lies in the plane $z = c$, the lift per unit area on the upper surface will be

$$-\frac{2\beta}{c^2} \cos \alpha \int_{z=0}^{c} \int_{r=0}^{z/\beta} (p_2 - p_1) \, dr dz.$$

Equations (25) and (26) indicate that p_3, the non-dimensional pressure, is a function of $\beta r/z$, α, ψ only. Thus if we make the transformation (4) into the cone-field coordinates (R, θ) and substitute for $p_2 - p_1$ by means of equation (15), one integration may be carried out, and the lift per unit area on the upper surface is

$$-(p_1 - p_0)\cos \alpha \int_0^1 p_3(R, \alpha, \psi) \, dR. \tag{29}$$

This integral occurs in most of the expressions for lift and drag coefficients, so it will be convenient at this point to calculate its value. Returning to the (ρ, θ) plane and the transformed upper half plane $\zeta = \xi + i\eta$ of section 3, the boundary condition at the point $\rho = 1$, $\theta = \alpha$, is that $p_3 = 0$ for $\alpha < \frac{1}{4}\pi + \frac{1}{2}\psi$, $p_3 = 2$ for $\frac{1}{4}\pi + \frac{1}{2}\psi < \alpha < \frac{3}{8}\pi + \frac{1}{4}\psi$, etc. Equation (26) shows that this is the value at $R = 1$, and hence, by a partial integration,

$$\int_0^1 p_3(R, \alpha, \psi)\, dR = -\int_{-\infty}^{-1} \frac{2\rho}{1+\rho^2} \frac{\partial p_3}{\partial \xi}\, d\xi,$$

for the case $\alpha < \frac{1}{4}\pi + \frac{1}{2}\psi$. For the remaining values of α, p_3 is not zero at $R = 1$, and hence there will be an extra term in this equation. The limits of the latter integral must be those points corresponding to $p_3 = 0$, $p_3 = 1$ on the wing, namely $\xi = -\infty$, -1. From the complex pressure integral (23), we have for the case $-\infty < \xi < -1$

$$\frac{\partial p_3}{\partial \xi} = \frac{\sin 2\tau}{\pi(\xi^2 - 1)^{\frac{1}{2}}(\xi^2 - \cos^2\tau)}.$$

Making the substitution $\xi = -\cosh\nu$, it follows from (26) that $e^\nu = \rho^{\pi/(\pi - 2\alpha)}$ and hence for the case $\alpha < \frac{1}{4}\pi + \frac{1}{2}\psi$ we have

$$\int_0^1 p_3(R, \alpha, \psi)\, dR = -\frac{1}{\pi} \int_0^\infty \frac{\sin 2\tau \, d\nu}{\cosh\{(\pi - 2\alpha)\nu/\pi\}[\cosh^2\nu - \cos^2\tau]} \tag{30}$$

with analogous results for other values. Values of this integral are given in Fig. 4 for positive and negative values of α, and for $\psi = 0°$, $20°$, $40°$, $60°$, $80°$. For the special case $\alpha = 0$, (30) reduces to

$$\int_0^1 p_3(R, 0, \psi)\, dR = -\frac{1}{\pi} \int_0^1 \frac{R^2 \sin 2\tau \, dR}{(1 - R^2)^{\frac{1}{2}}(1 - R^2 \cos^2\tau)} = -\tan\frac{1}{2}\psi. \tag{31}$$

Making use of the result (29), substituting for p_1 from equation (13), and using similar results for the lower face, the lift and drag coefficients are found to be:

$$C_L = \frac{2\epsilon \cos^2\alpha \tan\psi}{\beta}\left[2\operatorname{cosec}\psi + \int_0^1 p_3(R, \alpha, \psi)\, dR + \int_0^1 p_3(R, -\alpha, \psi)\, dR\right],$$

$$C_D = \frac{2\epsilon^2 \cos\alpha \tan\psi}{\beta}\left[2\operatorname{cosec}\psi + \int_0^1 p_3(R, \alpha, \psi)\, dR + \int_0^1 p_3(R, -\alpha, \psi)\, dR\right]. \tag{32}$$

In Fig. 5 values of $\beta C_L/\epsilon$ are plotted against $(\tan\lambda)/\beta$ for dihedrals of $0°$, $10°$, $30°$.
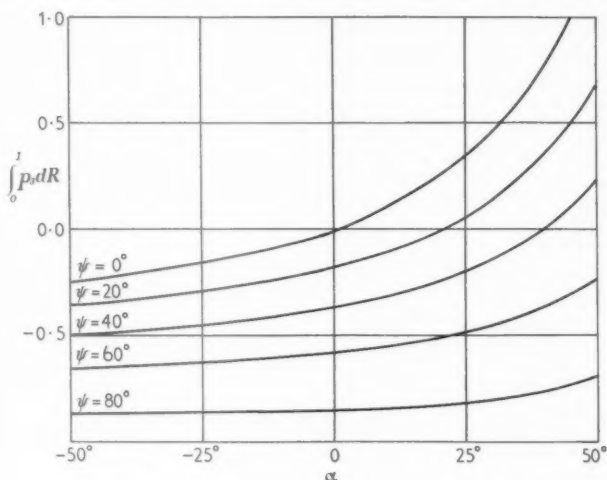
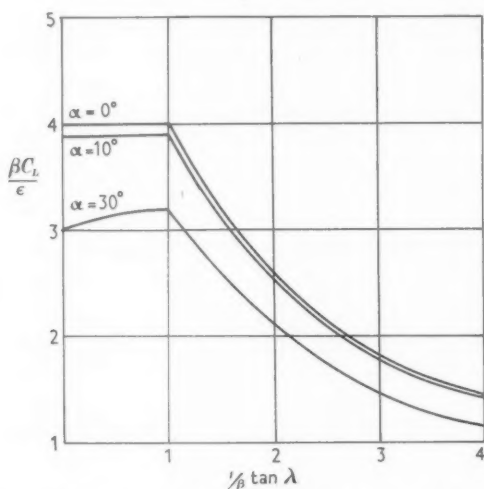FIG. 4. Variation of $\int\limits_0^1 p_3 \, dR$ with dihedral.

FIG. 5. Variation of the lift coefficient of a delta wing with sweepback and dihedral.

When there is no dihedral, the coefficients simplify by (31) to become

$$C_L = \frac{4\epsilon}{\beta}, \qquad C_D = \frac{4\epsilon^2}{\beta}, \tag{33}$$

which are the results previously obtained by Puckett (1) and Miss Roper (3).

5. Modified lift and drag coefficients

In order to obtain a solution it has been necessary to assume that the wings are of infinite span, and in particular the lift and drag coefficients determined in the previous section are also subject to this condition. Hence for wings of no sweepback no account is made of the effect of dihedral, because the constant pressure on the surface of the wing outside the Mach cone makes the only effective contribution to the force on the wing. These results are of little value for wings of finite span; it will therefore be convenient to introduce lift and drag coefficients that take into account only the modifying effects of dihedral. These coefficients will also be useful for determining the effects of dihedral on the lift and drag forces on wings of general cross-section.

It has been stated previously that the lift on the upper surface of the wing may be considered in two terms, that due to a pressure p_1 on the whole wing, and that due to a pressure $p_2 - p_1$ on a part of the wing. It is this last term which illustrates the effects of dihedral; hence we define the modified lift to be the lift produced by this pressure $p_2 - p_1$ acting on that part of the wing inside the Mach cone. The modified lift and drag coefficients C'_L and C'_D may then be defined in a similar manner, where the unit of area is taken to be the area of the wing inside the Mach cone. From the previous section, and in particular by substituting for $p_1 - p_0$ in equation (29), the modified lift and drag coefficients for a flat plate wing at incidence are

$$\left. \begin{aligned} C'_L &= \frac{2\epsilon \cos^2\alpha}{\beta \cos\psi} \left[\int_0^1 p_3(R, \alpha, \psi)\, dR + \int_0^1 p_3(R, -\alpha, \psi)\, dR \right] \\ C'_D &= \frac{2\epsilon^2 \cos\alpha}{\beta \cos\psi} \left[\int_0^1 p_3(R, \alpha, \psi)\, dR + \int_0^1 p_3(R, -\alpha, \psi)\, dR \right] \end{aligned} \right\} \quad (34)$$

Similarly, we can obtain the modified lift and drag coefficients for a wing of wedge-shaped cross-section. If the upper face has an incidence ϵ_1, and the lower face an incidence ϵ_2, so that the angle of the wedge is $\epsilon_2 - \epsilon_1$, then

$$\left. \begin{aligned} C'_L &= \frac{2\cos^2\alpha}{\beta \cos\psi} \left[\epsilon_1 \int_0^1 p_3(R, \alpha, \psi)\, dR + \epsilon_2 \int_0^1 p_3(R, -\alpha, \psi)\, dR \right] \\ C'_D &= \frac{2\cos\alpha}{\beta \cos\psi} \left[\epsilon_1^2 \int_0^1 p_3(R, \alpha, \psi)\, dR + \epsilon_2^2 \int_0^1 p_3(R, -\alpha, \psi)\, dR \right] \end{aligned} \right\} \quad (35)$$

The values of the integrals are given in Fig. 4, and for the case of a flat plate at varying angles of dihedral and sweepback, the modified lift coefficients are exhibited in Fig. 6.

Finally, it may easily be shown that for a wing of uniform cross-section, whose upper surface in the plane $\theta = \frac{1}{2}\pi$ is given by $r = f(z)$ and whose

lower surface in the plane $\theta = -\frac{1}{2}\pi$ is given by $r = g(z)$, the modified lift coefficient is

$$C_L' = -\frac{4\cos^2\alpha}{\beta c^2 \cos\psi}\left[\int_0^1 p_3(R,\alpha,\psi)\,dR \int_0^c f(z)\,dz - \int_0^1 p_3(R,-\alpha,\psi)\,dR \int_0^c g(z)\,dz\right],$$

$$(36)$$

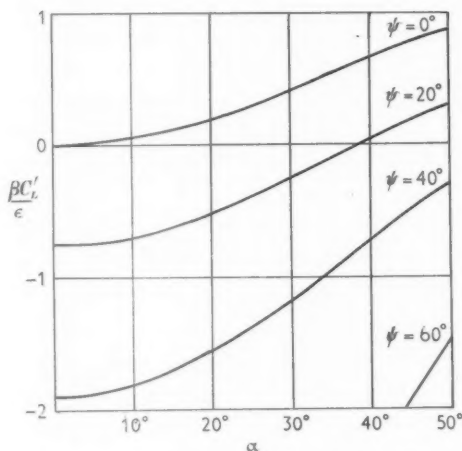where the trailing edge is assumed to lie in the plane $z = c$.

FIG. 6. Variation of the modified lift coefficient with dihedral.

6. The wing with the leading edge inside the Mach cone

The solution will be obtained for those wings whose upper and lower surfaces are flat, and possess incidences ϵ_1, ϵ_2 respectively. As before, we shall suppose the angle of sweepback to be λ, and the dihedral angle to be α.

Using the notation of section 2, there will be boundary conditions in the physical plane on the wings, implied by the condition of no normal velocity. Thus on the upper surface of the wing whose leading edge lies in the plane $\theta = \alpha$, $V_\theta = -U\epsilon_1\cos\alpha$, and on the lower surface of the same wing $V_\theta = -U\epsilon_2\cos\alpha$. Moreover the value of V_R in the plane $\theta = \frac{1}{2}\pi$ approaches $-U\epsilon_1$ as $r \to 0$, and similarly in the plane $\theta = -\frac{1}{2}\pi$ it approaches $+U\epsilon_2$ as $r \to 0$. There will also be the condition of symmetry that $V_\theta = 0$ in the planes $\theta = \pm\frac{1}{2}\pi$.

In the transformed (ρ, θ) plane given by (4), the wings may be approximated by the lines $\theta = \alpha$, $\theta = \pi - \alpha$. Also, the pressure is a solution of Laplace's equation (5), so that if we again denote by p_2 the pressure inside

F

the Mach cone, we may introduce a dimensionless pressure,

$$p_4 = \frac{p_2 - p_0}{\rho_0 U^2}, \tag{37}$$

which satisfies Laplace's equation and is zero on the unit circle $\rho = 1$.

Construct a circuit (Fig. 7) in the (ρ, θ) plane such that A is the point corresponding to the wing leading edge, AB is the lower surface of the
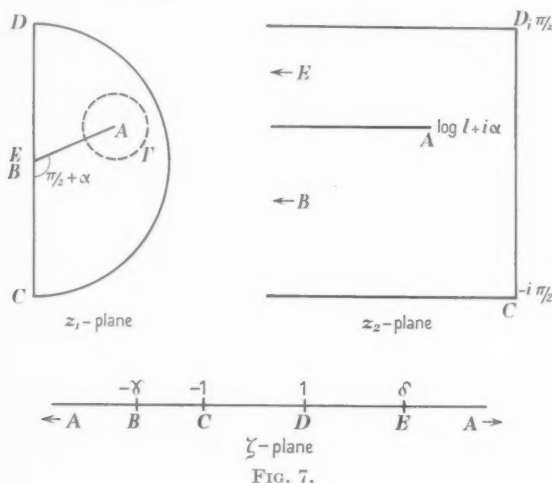

FIG. 7.

wing $\theta = \alpha$, AE is the upper surface, and CD is the unit circle $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. Then if A has coordinates (l, α) in the (ρ, θ) plane, so that $AB = l$,

$$\frac{2l}{1+l^2} = \beta \cot \lambda. \tag{38}$$

The boundary values of p_4 and its derivatives on the boundary consisting of the circuit $ABCDEA$, are found by substituting the conditions on V_R, V_θ in equations (11) and (12). Hence if we denote by $\partial/\partial s$ the derivative along this circuit, and by $\partial/\partial n$ the derivative along the inward normal to the circuit, the boundary conditions become

$$\left. \begin{array}{l} \dfrac{\partial p_4}{\partial s} = 0 \quad \text{on } CD \\[2mm] \dfrac{\partial p_4}{\partial n} = 0 \quad \text{on } AB,\, BC,\, DE,\, EA \end{array} \right\} . \tag{39}$$

It will be convenient to introduce cartesian coordinates x_1, y_1 in the (ρ, θ) plane so that $z_1 = x_1 + iy_1 = \rho e^{i\theta}$. We now map the configuration $ABCDEA$, in the z_1 plane, on to the upper half-plane by a transformation

employing the Schwarz–Christoffel theory. This was suggested by a similar transformation obtained by Lighthill (private communication), and defines the $\zeta = \xi + i\eta$ plane by the following transformations,

$$z_2 = \log z_1, \tag{40}$$

$$z_2 = \tfrac{1}{2}\pi i + K \int\limits_{1}^{\zeta} \frac{d\zeta}{(\zeta+\gamma)(\zeta-\delta)(\zeta^2-1)^{\frac{1}{2}}}, \tag{41}$$

where γ and δ are real and positive constants, and $(\zeta^2-1)^{\frac{1}{2}}$ is taken real and positive for ζ real and greater than 1. For suitable values of K, γ, δ this transforms conformally the z_1-plane configuration into the real axis of the ζ-plane, with $ABCDEA$ corresponding to $-\infty$, $-\gamma$, -1, 1, δ, ∞ respectively (see Fig. 7). Considering the change in the imaginary part of z_2 at E and B in equation (41), we obtain the relations

$$\tfrac{1}{2}\pi - \alpha = \frac{K\pi}{(\gamma+\delta)(\delta^2-1)^{\frac{1}{2}}},$$

$$\tfrac{1}{2}\pi + \alpha = \frac{K\pi}{(\gamma+\delta)(\gamma^2-1)^{\frac{1}{2}}}$$

from which we may eliminate K to obtain a relation between γ and δ

$$\frac{\pi-2\alpha}{\pi+2\alpha} = \frac{(\gamma^2-1)^{\frac{1}{2}}}{(\delta^2-1)^{\frac{1}{2}}}. \tag{42}$$

If we substitute for K, equation (41) may be integrated to obtain the transformation in the form

$$z_1 = i \left\{ \frac{(\gamma^2-1)^{\frac{1}{2}}(\zeta^2-1)^{\frac{1}{2}}+\gamma\zeta+1}{\zeta+\gamma} \right\}^{-(\pi+2\alpha)/2\pi} \left\{ \frac{(\delta^2-1)^{\frac{1}{2}}(\zeta^2-1)^{\frac{1}{2}}-\delta\zeta+1}{\zeta-\delta} \right\}^{(\pi-2\alpha)/2\pi}, \tag{43}$$

and hence, since as $z_1 \to le^{i\alpha}$, $\zeta \to \infty$, we have a second relation between γ and δ, namely

$$l = |(\gamma^2-1)^{\frac{1}{2}}+\gamma|^{-(\pi+2\alpha)/2\pi}|(\delta^2-1)^{\frac{1}{2}}-\delta|^{(\pi-2\alpha)/2\pi}. \tag{44}$$

Equations (42) and (44) are sufficient to define γ, δ for any given sweepback and dihedral (i.e. any given l and α); and equation (43) then gives the required transformation. In the ζ-plane the conditions (39) on the boundary become

$$\left.\begin{aligned} \frac{\partial p_4}{\partial \xi} &= 0 \quad \text{on the real axis for } |\xi| < 1 \\[2mm] \frac{\partial p_4}{\partial \eta} &= 0 \quad \text{on the real axis for } |\xi| > 1 \end{aligned}\right\}. \tag{45}$$

V_R and V_θ are bounded on the circuit $ABCDEA$ in the region of E and B,

even though they may possess discontinuities at these points. Thus equations (11) and (12) show that upon transforming to the ζ-plane

$$\lim_{h\to 0}\int_{-\gamma-h}^{-\gamma+h}\frac{\partial p_4}{\partial\eta}\,d\xi = 0, \qquad \lim_{h\to 0}\int_{-\gamma-h}^{-\gamma+h}\frac{\partial p_4}{\partial\xi}\,d\xi = 0, \qquad (46)$$

with a similar result holding at the point $\zeta = \delta$.

We define a complex function $W(\zeta)$ by

$$W(\zeta) = \frac{\partial p_4}{\partial\eta} + i\frac{\partial p_4}{\partial\xi}. \qquad (47)$$

It follows that $W(\zeta)$ is regular and integrable at all finite points in the upper half-plane, excepting $\zeta = \pm 1$ where it shall possess integrable singularities. Moreover, from equations (45) we have the conditions that on the real axis, $W(\zeta)$ is real for $|\xi| < 1$ and wholly imaginary for $|\xi| > 1$. But analogous problems suggest that at the wing leading edge, which corresponds to the point at infinity in the ζ-plane, the pressure should vary either logarithmically, or to the negative power of a half, with the distance in the physical plane from that edge. This indicates that for some constants A and B, $W \sim i(A+B/\zeta)$ as $\zeta \to \infty$. Consider the integral $\int_k^{-k} W(\zeta)\,d\zeta$ whose path of integration in the ζ-plane consists of a large semicircle of radius k. It is shown in the appendix that by transforming to the z_1-plane and using the boundary conditions on V_R, V_θ at the surface of the wing,

$$\mathrm{re}\left\{\lim_{k\to\infty}\int_k^{-k} W\,d\zeta\right\} = -\frac{2l\cos\alpha}{\beta(1-l^2)}(\epsilon_2-\epsilon_1). \qquad (48)$$

A function $W(\zeta)$ which satisfies all these conditions is

$$W(\zeta) = i\frac{A\zeta+B}{(\zeta^2-1)^{\frac{1}{2}}}, \qquad (49)$$

where in this expression A and B are real constants. Equation (48) gives an additional condition which determines B; thus we have

$$B = \frac{2l\cos\alpha}{\pi\beta(1-l^2)}(\epsilon_2-\epsilon_1), \qquad (50)$$

and integrating $W(\zeta)$ we obtain the pressure on the surface of the wing from

$$p_4 = A(\xi^2-1)^{\frac{1}{2}} + B\log|\xi+(\xi^2-1)^{\frac{1}{2}}|. \qquad (51)$$

If we consider the upper surface of the wing alone, the transformation (43) may be considerably simplified by writing

$$\xi = \cosh a, \qquad \gamma = \cosh b, \qquad \delta = \cosh c \qquad (52)$$

so that a is real and positive on that part of the real axis for which $\xi > 0$. The relations (42) and (44) then become

$$(\pi + 2\alpha)\sinh b - (\pi - 2\alpha)\sinh c = 0, \tag{53}$$

$$(\pi + 2\alpha)b + (\pi - 2\alpha)c + 2\pi \log l = 0, \tag{54}$$

and the transformation reduces to

$$z_1 = i \left\{ \frac{1 - \tanh \tfrac{1}{2}a \tanh \tfrac{1}{2}b}{1 + \tanh \tfrac{1}{2}a \tanh \tfrac{1}{2}b} \right\}^{(\pi+2\alpha)/2\pi} \left\{ \frac{\tanh \tfrac{1}{2}c - \tanh \tfrac{1}{2}a}{\tanh \tfrac{1}{2}c + \tanh \tfrac{1}{2}a} \right\}^{(\pi-2\alpha)/2\pi}. \tag{55}$$

Hence we may write (51) in the form

$$p_4 = A \sinh a + B a. \tag{56}$$

The constant B is given by (50), but to evaluate A, we use equation (9) to give an expression for V_R on the line $\theta = \tfrac{1}{2}\pi$. Including the factor $\rho_0 U^2$ introduced by equation (37) we have, since V_R is zero on the unit circle

$$V_R = \frac{\beta U}{2} \int_1^\rho \frac{1 + \rho^2}{\rho} \frac{\partial p_4}{\partial \rho} \, d\rho.$$

The remaining boundary condition on the upper surface is that $V_R \to -U\epsilon_1$ as $\rho \to 0$ along $\theta = \tfrac{1}{2}\pi$. Hence, transforming the integral to the ζ-plane, where the integral is taken along the real axis, we have, using (52) and (56),

$$-\frac{2\epsilon_1}{\beta} = \int_0^c \frac{1 + \rho^2}{\rho} [A \cosh a + B] \, da, \tag{57}$$

where ρ is given by

$$\rho = \left\{ \frac{1 - \tanh \tfrac{1}{2}a \tanh \tfrac{1}{2}b}{1 + \tanh \tfrac{1}{2}a \tanh \tfrac{1}{2}b} \right\}^{(\pi+2\alpha)/2\pi} \left\{ \frac{\tanh \tfrac{1}{2}c - \tanh \tfrac{1}{2}a}{\tanh \tfrac{1}{2}c + \tanh \tfrac{1}{2}a} \right\}^{(\pi-2\alpha)/2\pi}. \tag{58}$$

7. Lift and drag coefficients for a thin delta wing lying entirely inside the vertex Mach cone

For a thin wing at incidence ϵ, we have, in the previous notation, $\epsilon_1 = \epsilon_2 = \epsilon$ so that by (50), the constant $B = 0$. Thus the results obtained for the pressure on the wing may be simplified to

$$p_4 = A \sinh a, \tag{59}$$

where the constant $A = A(l, \alpha)$ is determined by

$$\frac{2\epsilon}{\beta A} = - \int_0^c \frac{1 + \rho^2}{\rho} \cosh a \, da \tag{60}$$

and ρ is given by equation (58). In its present form the above equation cannot be integrated in terms of elementary functions, hence the

pressure is not explicitly given, but for two special instances the pressure may be simplified.

The first case arises when there is no dihedral, so that $\alpha = 0$. The expression for A may then be integrated, since $b = c$, by the substitution $\sinh a = \sinh b \cos \mu$, and

$$A = -\frac{4\epsilon l^2}{\beta(1-l^4)E(q)}, \tag{61}$$

where $E(q)$ is the complete elliptic integral of the second kind of modulus $q = (1-l^2)/(1+l^2)$ and l is given by (38). Clearly, equation (55) may now be simplified to give an expression for $\sinh a$ in terms of ρ, for points on the upper surface of the wing $\theta = \alpha$. Hence the pressure on the upper surface of the wing will be given in the physical polar coordinates r, θ, z by

$$p_4 = -\frac{\epsilon \cot^2\lambda z}{E(q)(z^2 \cot^2\lambda - r^2)^{\frac{1}{2}}}. \tag{62}$$

In this expression, as in the previous sections, λ represents the sweepback of the wing. This result for the pressure was previously obtained by Robinson (8) and Stewart (2).

The other case of interest occurs when the leading edge approaches the vertex Mach cone from the inside. This arises as $(\tan\lambda)/\beta \to 1$, and hence from equation (38), by the limit as $l \to 1$. Suppose, then, that $l = e^{-t}$ where t is small. Equations (53) and (54) indicate that to the first order in t

$$b = \frac{\pi t}{\pi + 2\alpha}, \qquad c = \frac{\pi t}{\pi - 2\alpha}.$$

Hence as $t \to 0$ both b and c approach zero, and for the purposes of integrating to determine the constant A, we may take a to be small. Thus

$$\rho = \left\{\frac{\pi t - (\pi - 2\alpha)a}{\pi t + (\pi - 2\alpha)a}\right\}^{(\pi-2\alpha)/2\pi}. \tag{63}$$

By making the substitution $\rho = e^{\mu}$ and using the above result we have

$$A = -\frac{2\epsilon \cos \alpha}{\pi t \beta}. \tag{64}$$

The expression (55) gives values of ρ for those points lying on the upper surface of the wing $\theta = \alpha$. Since b and c are first-order expressions in t, it follows that a is also of the first order at all points of the wing apart from the leading edge. Hence we may rewrite (63) to give $\sinh a$ in terms of ρ, and the pressure distribution on the wing is then obtained from

$$p_4 = -\frac{2\epsilon \cos \alpha}{\beta(\pi - 2\alpha)} \left\{\frac{1 + \rho^{2\pi/(\pi-2\alpha)}}{1 - \rho^{2\pi/(\pi-2\alpha)}}\right\}. \tag{65}$$

Substituting for p_4 in (37), it will be seen that this result agrees with

equation (28) obtained by the limit as the leading edge approaches the vertex Mach cone from the outside.

Returning to the general case, we obtain the lift and drag coefficients by first considering the normal force on that part of the wing $\theta = \alpha$, lying between the planes $z = 0$ and $z = \beta$. If we omit the term due to the pressure at infinity, p_0, and write p_4, the non-dimensional pressure, as $p_4(\rho, \alpha, l)$, the normal force on the upper surface is

$$-\rho_0 U^2 \int_{z=0}^{z=\beta} \int_{r=0}^{r=z\cot\lambda} p_4(\rho, \alpha, l)\, dr dz.$$

This may be integrated once by making the substitution (4), viz. $\dfrac{\beta r}{z} = \dfrac{2\rho}{1+\rho^2}$ and replacing p_4 by the form given in equation (59). A similar result will hold for the lower surface, and if we denote by ρ' the appropriate value for ρ corresponding to the lower surface at $\zeta = -\cosh a$, the solutions for the lift and drag coefficients are

$$C_D = \epsilon C_L = -\frac{4\epsilon A}{\beta}\cos\alpha\tan\lambda\left[\int_0^l \frac{1-\rho^2}{(1+\rho^2)^2}\sinh a\, d\rho + \int_0^l \frac{1-\rho'^2}{(1+\rho'^2)^2}\sinh a\, d\rho'\right].$$
(66)

Define a_1, b_1, c_1, such that $\tanh\frac{1}{2}a = a_1$, $\tanh\frac{1}{2}b = b_1$, $\tanh\frac{1}{2}c = c_1$ and ρ and ρ' are given by

$$\rho = \left\{\frac{1-a_1 b_1}{1+a_1 b_1}\right\}^{(\pi+2\alpha)/2\pi}\left\{\frac{a_1-c_1}{a_1+c_1}\right\}^{(\pi-2\alpha)/2\pi},$$

$$\rho' = \left\{\frac{1-a_1 c_1}{1+a_1 c_1}\right\}^{(\pi-2\alpha)/2\pi}\left\{\frac{a_1-b_1}{a_1+b_1}\right\}^{(\pi+2\alpha)/2\pi},$$

then the integrals occurring in equation (66) may be simplified for numerical integration by the following substitution, which removes the singularity at $\rho = l$,

$$\int_0^l \frac{1-\rho^2}{(1+\rho^2)^2}\sinh a\, d\rho = 2\int_{c_1}^1 \left[\frac{l}{1+l^2} - \frac{\rho}{1+\rho^2}\right]\frac{1+a_1^2}{(1-a_1^2)^2}\, da_1 - \frac{2lc_1}{(1+l^2)(1-c_1^2)}.$$

The value of the integrand in the right-hand integral is finite at $a_1 = 1$ and equal to

$$\frac{\pi+2\alpha}{2\pi}\frac{l(1-l^2)b_1(1-b_1^2 c_1^2)}{(1+l^2)^2(1-b_1^2)^2(1-c_1^2)}.$$

Hence from (66) the values of C_L and C_D may be computed, the constant A being found by the integral (60), where the singularity at $a = c$ should first be removed. Values of $\beta C_L/\epsilon$ are plotted against $(\tan\lambda)/\beta$, for $\alpha = 0°$, $10°$, $30°$ in Fig. 5.

8. Conclusion

The solutions obtained are particularly interesting when applied to the lift coefficient of a delta wing. If the interference effect of the two wings were to be neglected, we would expect the lift coefficient for a dihedral α to be a factor $\cos^2\alpha$ times the lift coefficient for zero dihedral. This follows because the incidence ϵ is measured in a vertical plane making the effective incidence of the wing $\epsilon\cos\alpha$, and because resolving the force on the wing in a vertical direction introduces another factor $\cos\alpha$. The results shown in Fig. 5 substantiate this intuitive approach for dihedral angles less than $30°$, and it is for the purpose of obtaining the degree of accuracy with which this simple rule may be applied that a table is given below of values of $\{C_L(\alpha)\sec^2\alpha-C_L(0)\}/C_L(0)$, where $C_L(\alpha)$ is the lift coefficient for a particular dihedral angle α.

TABLE 1

$$Values\ of\ D(\alpha) = \frac{C_L(\alpha)\sec^2\alpha-C_L(0)}{C_L(0)}\ for\ \alpha = 10°,\ 30°$$

$\frac{1}{\beta}\tan\lambda$	$D(10°)$	$D(30°)$	$\frac{1}{\beta}\tan\lambda$	$D(10°)$	$D(30°)$
0·000	0·000	0·000	1·143	0·006	0·073
0·342	0·003	0·035	1·333	0·006	0·075
0·643	0·005	0·053	2·000	0·006	0·074
0·866	0·006	0·062	4·000	0·003	0·067
1·000	0·006	0·068	∞	0·000	0·000

From this table it is clear that the maximum deviation from the $\cos^2\alpha$ rule occurs at such a sweepback that the leading edge lies just inside the vertex Mach cone. The error for a dihedral angle of $10°$ is less than $0\cdot7$ per cent., whilst for a dihedral of $30°$ it is less than 8 per cent. As the linearized theory may be no more accurate than this in most practical applications, the lift coefficient may be calculated by the above rule for dihedral angles less than $30°$, without serious error.

Since this paper was begun, the delta wing with dihedral has been examined by Nocilla (9) for the case when the leading edge lies inside the Mach cone. By expanding the result as a power series to the fourth power in the dihedral angle he obtains the pressure on the wing surface. The first terms agree immediately with the results given here, but unfortunately as no graphs were given, and as the solutions obtained in this paper are not readily amenable to expansion in a power series of the required form, further terms have not been compared.

The yawed dihedral wing with the leading edges outside or inside the Mach cone of the vertex will be considered in another paper. Values in

sideslip, of the rolling moment and other derivatives will be obtained from the results for this case.

In conclusion, I should like to thank Dr. W. Chester for suggesting the subject of this paper, and for a number of useful suggestions regarding it. I am also indebted to the Department of Scientific and Industrial Research for a maintenance grant.

APPENDIX

To prove that if $W(\zeta)$ is a function defined in the $\zeta = \xi + i\eta$ plane by

$$W = \frac{\partial p_4}{\partial \eta} + i \frac{\partial p_4}{\partial \xi}$$

then

$$\mathrm{re}\left\{\lim_{k \to \infty} \int_{k}^{-k} W \, d\zeta\right\} = -\frac{2l\cos\alpha}{\beta(1-l^2)}(\epsilon_2 - \epsilon_1).$$

Consider in the z_1-plane of section 6 the integral

$$I = \int_{\Gamma} \left(\frac{\partial p_4}{\partial y_1} + i\frac{\partial p_4}{\partial x_1}\right) dz_1,$$

where Γ is a circle of radius σ (where σ is small) whose centre is the point A (see Fig. 7). The circuit begins on the upper surface and ends on the lower surface of the wing $\theta = \alpha$. If this integral be transformed into polar coordinates we have

$$I = \int_{\Gamma} \left(\frac{1}{\rho}\frac{\partial p_4}{\partial \theta}\, d\rho - \rho \frac{\partial p_4}{\partial \rho}\, d\theta\right) + i \int_{\Gamma} \left(\frac{\partial p_4}{\partial \rho}\, d\rho + \frac{\partial p_4}{\partial \theta}\, d\theta\right).$$

We may now substitute for $\frac{1}{\rho}\frac{\partial p_4}{\partial \theta}$, $\frac{\partial p_4}{\partial \rho}$ by equations (11) and (12), where we include the factor $\frac{1}{\rho_0 U^2}$ to account for the non-dimensional pressure introduced by equation (37). If the change in value whilst completing the circuit Γ is denoted by $[\ ]_\Gamma$ and approximations are made for $\rho/(1-\rho^2)$ on the circle Γ the integral becomes

$$I = \frac{2l}{\beta U(1-l^2)}[V_\theta]_\Gamma + i[p_4]_\Gamma + O\{\sigma(V_R^2 + V_\theta^2)^{\frac{1}{2}}\}.$$

But on the assumption that any singularities of the velocity are $O(\sigma^{-1})$, the limit as $\sigma \to 0$ exists. Hence substituting for the boundary values of V_θ on the wing we have

$$\lim_{\sigma \to 0} I = -\frac{2l\cos\alpha}{\beta(1-l^2)}(\epsilon_2 - \epsilon_1) + i[p_4]_\Gamma$$

which upon transforming to the ζ-plane gives the required result.

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THEORY OF WINDING IN CENTRIFUGE SPINNING

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SUMMARY

Equations of motion in the presence of air-drag are found for a string or 'yarn' which is being unwound from the inside of a very rapidly rotating cylinder and wound on to a stationary cylinder of smaller diameter. The equations are solved, the conditions under which steady winding can be obtained are determined, the existence of physically possible solutions† within these limits is proved and a formula for the tension is given.

1. Introduction

IN the Centrifuge Spinning System (1) a hollow cylinder (called a 'pot') is continuously rotated at high speed and yarn is first fed down a tube concentric with the axis of rotation, accumulating on the inner wall of the pot and being held there by centrifugal force and friction. When sufficient yarn has accumulated the feed is stopped and a stationary cylinder is inserted concentrically inside the pot and the yarn first engages this then winds on to it. Problems arising in this winding process have led to the mathematical analysis given in this paper, and this provides answers which, obtained by experimental methods, would require much labour, expense, and time. The conditions under which winding is steady are an example, for the analysis shows that it is air-drag which makes the winding steady (*in vacuo* it is necessarily unstable), that the ratio of inner cylinder radius to outer (winding) radius must never exceed 0·5 and that this value is possible only if air-drag is suitably related to the mass per unit length of the yarn and the dimensions of the system.‡

Some assumptions had, of course, to be made about air-drag and they are (i) that it is normal to each element of the yarn, and (ii) that it is proportional to the square of the component of air velocity along this normal. Published measurements of air-drag on textile yarns show that this is so to a good approximation (2). The yarn is also assumed to be flexible and inextensible but not weightless, though since centrifugal forces are very large, gravity is negligible and thus the problem is two-dimensional.

With these assumptions the equations of motion have a solution in terms of Bessel functions and from the properties of these functions the main features of the centrifuge winding can be determined, and, if the accuracy required is not too great, with little computation.

† There are certain solutions which are mathematically real but not physically possible.
‡ In fact a model on a smaller scale has different properties.

2. The equations of motion

Fig. 1 shows a diagram of the yarn as it is being wound on to a stationary cylinder of radius a from its position on the inside of a cylinder of radius b which rotates with angular velocity ω_1. Variation of a and b with time is small compared with other effects and they can be regarded as virtually constant. Hence the yarn adopts a curve, somewhat as shown in Fig. 1,
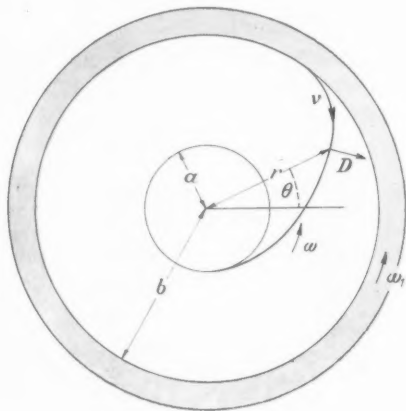

FIG. 1

which rotates with apparent angular velocity ω. Coriolis forces of some magnitude exist, for the yarn moves along this curve with a (virtually) constant velocity v and, by considering the rate of winding on the inner cylinder $r = a$, we see that

$$v = a\omega. \tag{1}$$

Considering the rate of unwinding at $r = b$, we see that

$$(b-a)\omega = b\omega_1. \tag{2}$$

The air-drag, $D\,ds$, on an element ds of the yarn curve is, by assumption, normal to the curve but we shall not specify its magnitude until later.

We denote the tension of the yarn at any point by T, its mass per unit length (assumed constant) by m, and derivatives with respect to s, the length along the curve, by dots. We may now write down the equations of motion.

The only acceleration on an element of yarn which has a component along the tangent to the curve is the centrifugal acceleration, and resolving tangentially, we find that

$$\dot{T}+m\omega^2r\dot{r} = 0. \tag{3}$$

Equating the moment about the axis of forces on an element to the rate of change of angular momentum, we see that

$$d(Tr^2\dot\theta)/ds - Dr\dot r = mv\,d(r^2\dot\theta)/dt - 2m\omega r\,(dr/dt)$$
$$= mv^2\,d(r^2\dot\theta)/ds - 2m\omega vr\dot r$$

and by using the relation (1), this may be written in the form

$$d[(T - mv^2)r^2\dot\theta]/dr = (D - 2ma\omega^2)r. \qquad (4)$$

Note also that

$$\dot r^2 + r^2\dot\theta^2 = 1. \qquad (5)$$

By (3) the tension in the string satisfies

$$T + \tfrac12 m\omega^2 r^2 = \text{constant}$$

which we shall write as

$$T - mv^2 = \tfrac12 m\omega^2[h^2 - r^2], \qquad (6)$$

where h is an arbitrary constant determined by boundary conditions. These conditions are, if there is no abrupt change in direction as the yarn unwinds at $r = b$ and winds on at $r = a$, that

$$r\dot\theta = 1, \qquad r = a \text{ or } b. \qquad (7)$$

Substituting from (6) we find that (4) becomes

$$d[m\omega^2(h^2 - r^2)r^2\dot\theta]/dr = 2(D - 2ma\omega^2)r. \qquad (8)$$

3. Solution for zero air-drag

When $D \equiv 0$ integration of (8) gives

$$r\dot\theta = 2a[k^2 - r^2]/[r(h^2 - r^2)] \qquad (9)$$

where k is arbitrary. From (7) we find that

$$h^2 = b^2 - ab - a^2, \qquad k^2 = b(b - a)/2.$$

Hence, as r increases from a to b, $h^2 - r^2$ vanishes and, by (9), $r\dot\theta$ becomes infinite, which is impossible by (5). No steady curve can exist in these circumstances. However, the presence of air-drag may possibly stabilize the system and we now investigate the circumstances in which it does so.

4. Solution when air-drag is proportional to normal velocity squared

The air velocity at the element ds is $r\omega$ (see Fig. 1) and its component in the direction of the normal is $r\omega\dot r$. Hence we shall take

$$D = G\omega^2 r^2\dot r^2 \qquad (10)$$

where G is a constant of proportionality. Since, by (5),

$$D = G\omega^2(r^2 - r^4\dot\theta^2)$$

it will be seen that (8) and (10) combine to produce a generalized Riccati equation which can be solved by the method described by Watson (3), p. 92.

Introducing the following dimensionless parameters

$$r/h = R, \quad a/h = A, \quad b/h = B, \quad Gh/m = \mu, \quad x = 2\mu(1-R^2)^{\frac{1}{2}},$$
(11)

the Riccati equation is

$$d[x^2 R(r\dot\theta)]/d(x^2) = 2A - \mu + x^2/(4\mu) + \mu R^2(r\dot\theta)^2$$

the solution of which is, using transformation (2) on p. 92 of (3) and equations (3) and (4), ibid., p. 97,

$$2\mu R(r\dot\theta) = 1 - x[J_\nu'(x) + MY_\nu'(x)]/[J_\nu(x) + MY_\nu(x)],$$
(12)

$$\nu^2 \equiv 1 - 8\mu A + 4\mu^2,$$
(13)

$$r\dot\theta = 1, \quad R = A \text{ or } B,$$
(14)

where J_ν and Y_ν are Bessel functions of the first and second kind, J_ν' and Y_ν' are their derivatives with respect to x, and M is an arbitrary constant.

The determination of h and M from the boundary conditions (7) might appear to involve much numerical work, especially when ν is non-integral, but a study of the properties of the solution (12) enables practical answers to be obtained with very little computation.

5. Properties of the solution

We shall, initially, consider ν to be fixed, treating μ and A as variables which, however, must satisfy (13). We further find it convenient to divide our investigation into two parts (a) $\nu \geqslant 1$, and (b) $\nu < 1$.

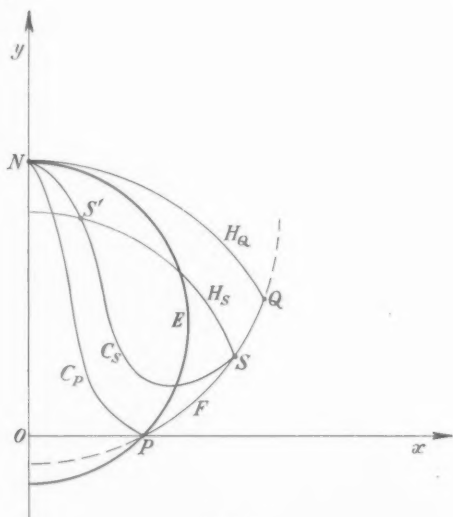(a) *Fixed* $\nu \geqslant 1$

FIG. 2

We first examine the properties of the right-hand side of (12); this leads us to consider what we shall call C-curves, whose equation is

$$C \equiv y = 1 - x[J'_\nu(x) + MY'_\nu(x)]/[J_\nu(x) + MY_\nu(x)]. \tag{15}$$

In Fig. 2, C_s and C_P represent two such curves. From the differential equation satisfied by Bessel functions it can be seen that a C-curve satisfies

$$x \, dy/dx = (y-1)^2 + x^2 - \nu^2. \tag{16}$$

Hence, outside the circle E, where

$$E \equiv (y-1)^2 + x^2 = \nu^2 \tag{17}$$

all C-curves have slopes of the same sign as x, but inside E slopes of the reverse sign. Hence if a C-curve cuts E it does so at a minimum, unless $x = 0$ at this point. Since minima and maxima alternate, a C-curve which cuts E when $x \neq 0$ must cut E again when $x = 0$ and have a maximum there. This second point of intersection must be the point $x = 0, y = 1+\nu$, N say. If a C-curve cuts E at the other point for which $x = 0$, namely, $x = 0, y = 1-\nu$ then, by the above reasoning it must have a minimum there; in fact by examining equation (15) we see that for this C-curve, $M = 0$ since $Y_\nu(x)$ is of the order of $x^{-\nu}$ as $x \to 0$ and hence if $M \neq 0$ the right-hand side of (15) $\to 1+\nu$ as $x \to 0$. It should be noted that, for certain values of M, the corresponding C-curve cuts E at N only and has a minimum there (the curve C' of Fig. 3 is an example).

Examination of the left-hand side of (12) when $r\dot\theta = 1$ leads us to consider what we shall call H-curves which have the parametric form

$$x = 2\mu(1-R^2)^{\frac12}, \qquad y = 2\mu R$$

and, consequently, the explicit form

$$H \equiv x^2 + y^2 = 4\mu^2. \tag{18}$$

At the intersection of a C- and an H-curve we have a solution of (12) with $r\dot\theta = 1$, though whether the corresponding value of R can be made equal to A so as to satisfy the first boundary condition of (14) depends on whether (13) is satisfied. This will not be so in general but will be the case if the intersection lies on what we shall call the F-curve which has the parametric equations

$$x = 2\mu(1-A^2)^{\frac12}, \qquad y = 2\mu A, \qquad \nu^2 = 1 - 8\mu A + 4\mu^2. \tag{19}$$

Eliminating μ and A we find that

$$F \equiv (y-2)^2 + x^2 = \nu^2 + 3. \tag{20}$$

Note that if y is positive F lies outside E and that E and F intersect at P where $x = \sqrt{(\nu^2-1)}, y = 0$.

Now take a point S, with coordinates x_s, y_s, say, on the F-circle and

consider the C- and H-curves through S, C_s, and H_s say. We shall show
(i) that if S lies on the arc PQ of F where Q is the point

$$x = \sqrt{3}(1+\nu)/2, \qquad y = (1+\nu)/2,$$

then C_s, H_s provide a complete set of solutions of (12), (13), and (14), and
(ii) there are no other solutions.

To do this we first prove the following results:

(i) No two C-curves intersect, hence C_s is unique.

(ii) C_s crosses F from inside to outside as x increases.

(iii) The C-curve through P, C_P say, has a positive ordinate for $0 \leqslant x \leqslant$ greatest abscissa of F, except at P where the ordinate is zero.

(iv) If $x > 0$, C_P cannot cut F other than at P.

(v) If $x > 0$ no C-curve can cut F twice.

(vi) For $0 < x < x_s$ the ordinate of C_s is greater than the ordinate of C_P (unless $S \equiv P$) and, hence, by (iii) is positive in this range.

(vii) H_s cuts C_s at at least one other point, S' say, for which $0 < x < x_s$

Now if two C-curves intersect then their slopes, by (16), must be equal at the point of intersection and, hence, since (16) is a first-order differential equation, the curves coincide everywhere. Thus (i) is proved.

Since the arc PQ lies outside E, then C_s has a positive slope at S by (16); and, hence, to prove (ii) we must show that the slope of F at S is either negative or, if positive, greater than that of C_s. Now the slope of F at S is, by (20), $x_s/(2-y_s)$ and that of C_s, by (16) and (20), is $2y_s/x_s$. But since x_s, y_s satisfy (20),

$$x_s^2 = \nu^2 - 1 + 4y_s - y_s^2 \geqslant 2y_s(2-y_s) \tag{21}$$

as $\nu^2 \geqslant 1$ and $y_s \geqslant 0$, equality only holding if $\nu = 1$, $y_s = 0$. Hence (ii) follows. It is worth noting that, for fixed ν, the maximum value of y_s occurs when $S \equiv Q$ and is $(1+\nu)/2$; so that for $\nu < 3$, $y_s < 2$ but for $\nu > 3$, y_s can be greater than 2 and F can thus have a negative slope at S.

Since P is a point on E, C_P has a minimum at P and thus must pass through N. Hence for $0 \leqslant x < \sqrt{3}(1+\nu)/2$, C_P has a positive ordinate. For $x > \sqrt{3}(1+\nu)/2$ and increasing, C_P has a positive slope by (16) and, hence a positive ordinate until an infinity is reached. However C_P cannot cut F other than at P, for C_P would cross F from outside to inside as x increased, which contradicts (ii). Hence (iii) and (iv) are proved while a similar argument proves (v).

Now, if $S \not\equiv P$, $y_s > 0$, and by (iv), is greater than the ordinate of C_P for $x = x_s$. Hence (vi) follows for, if not, C_s must cross C_P and contradict (i).

To prove (vii) we trace out the paths of C_s and H_s starting at S, i.e. at

$x = x_s$, and let x decrease. H_s is a circle with centre the origin O, and its ordinate must therefore increase. The ordinate of C_s must decrease initially since C_s has a positive slope outside E. When E is reached, C_s has a minimum, though by (vi) its ordinate is still positive. As x decreases further C_s increases until at $x = 0$ it reaches N. But by (18) if the radius of H_s is less than $(1+\nu)$, H_s cuts the y-axis at a point below N and therefore there is at least one point S' satisfying (vii). If there is more than one such point of intersection of H_s and C_s we shall call that point with the largest abscissa S'.

The ratio of the ordinates of C_s and H_s between S and S' gives the value of $r\dot\theta$ as can be seen from (12). This ratio is positive but less than unity and hence, by (5), $\dot r$ has a real value and the solution is physically possible. At S', $r\dot\theta = 1$ and the corresponding value of R is a value of B satisfying the second condition of (14). The circle H_Q where

$$H_Q \equiv x^2 + y^2 = (1+\nu)^2 \tag{22}$$

gives the maximum permissible H-circle and this cuts the F-curve at Q. For H-circles with larger radii there is no point S' and so no complete solution of (12), (13), and (14).

The reader may wonder what a second intersection of C_s and H_s, S'' say, would imply. Now, at S', C_s and H_s would cross, in general, and so their ratio (and hence $r\dot\theta$) would be greater than unity between S' and S'' and no physically possible solution would exist in this region. Similarly, if S lay on the F-curve below P (i.e. with a negative ordinate) the value of A and, hence, of a, the inner radius, would be negative which is physically impossible.

We shall conclude this subsection with a proof that A/B is always less than or equal to 0.5. Consider the point, X, Y say, on each H_s circle with ordinate twice that of S. By (19) we may put $y_s = 2\mu A$ and thus, by (8), $X = 2\mu(1-4A^2)^{\frac{1}{2}}$, $Y = 4\mu A$. Eliminating μ and A between these relations and (13), we find that

$$(Y-1)^2 + X^2 = \nu^2$$

the equation of E! But S' lies inside E unless $2\mu = (1+\nu)$ when $A = 0.5$ and $B = 1.0$ and so it follows that $A/B \equiv a/b$ must be less than 0.5.

Numerical calculations show that B is close to unity if $\nu \leqslant 2$ as can be seen from Table 1

TABLE 1

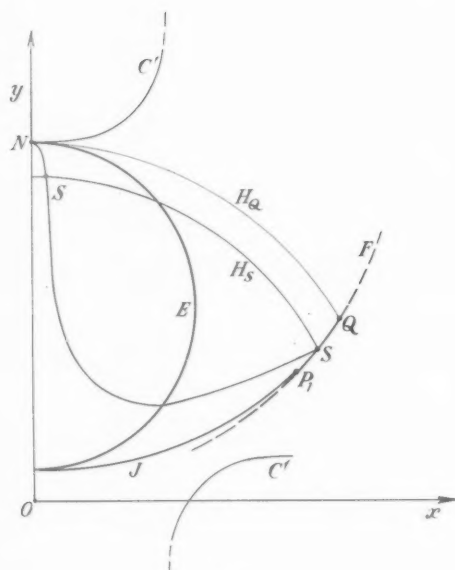| ν	1·0	1·1	1·2	1·3	1·4	1·5	2·0	3·0	4·0
Minimum value of B	0·98	0·95	0·95	0·94	0·93	0·92	0·89	0·85	0·81

(b) *Fixed* $\nu < 1$

FIG. 3

Here the F-curve no longer cuts E. The lower point on the permissible arc of F corresponding to real solutions we have marked P_1 in Fig. 3. P_1 is in fact the intersection of F and the curve J where

$$J \equiv y = 1 - x\,J'_\nu(x)/J_\nu(x)$$
$$= 1 - \nu + x^2/[2(\nu+1)] + x^4/[8(\nu+1)^2(\nu+2)] + \dots \quad (23)$$

this expansion being given by Rayleigh (4). This follows from the fact that J cuts E at the point $x = 0$, $y = 1-\nu$ and any C-curve with an ordinate lower than J for the same abscissa will be connected to N (since its value of M in (12) is non-zero) via an infinity, as illustrated by C' in Fig. 3. Hence for real solutions S must lie on the arc $P_1 Q$. Again H_Q is still the upper limiting H-circle and the existence of a physically possible solution follows by the same argument as for $\nu \geqslant 1$.

If the coordinates of P_1 are $x_1 \equiv 2\mu_1(1-A_1^2)^{\frac{1}{2}}$, $y_1 = 2\mu_1 A_1$, then x_1, y_1 must satisfy (23) and μ_1, A_1 must satisfy (13). Hence we find that

$$A_1 = \mu_1/2 + \mu_1^3/24 + O(\mu_1^5), \quad (24)$$
$$\mu_1 = [6(1-\nu)]^{\frac{1}{4}} + O(1-\nu)^{\frac{3}{4}}. \quad (25)$$

Numerical calculation shows that when $\nu = 0.87$, then P_1 and Q coincide and that below this value of ν there are no solutions of (12), (13), and (14).

6. Conclusions

We have shown that the maximum possible ratio of $A/B \equiv a/b$ is 0·5 and that this is only attainable if $\mu \equiv Gh/m$ is greater than a certain value, namely, if $2\mu > 1\cdot87$. If μ is lower than this value then it is not possible to make A as large as 0·5 but the maximum value of A will be given by (24), i.e. by

$$A = \mu/2 + \mu^3/24 + \dots. \tag{26}$$

For textile yarns G/m lies between 0·04 and 0·22 (see (**2**)) and since B is close to unity we may take $B \doteq 1$, i.e. $h \doteq b$ for purposes of computing μ approximately. Again the approximation $B \doteq 1$ enables us to specify the tension at the inner cylinder $r = a$. It is a little greater than

$$\tfrac{1}{2}m\omega^2(b^2 + a^2)$$

as can be seen from (1) and (6).

More accurate values of B can be obtained from a prepared table, whence μ and hence the maximum ratio of $A/B \equiv a/b$ can be calculated accurately.

The shape of curve adopted by the yarn is less important but may be determined numerically from the formula (12) for $r\dot{\theta}$.

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ON THE THEORY OF INITIALLY TENSIONED CIRCULAR MEMBRANES SUBJECTED TO UNIFORM PRESSURE

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SUMMARY

The solution of the problem of a tightly stretched membrane subjected to uniform pressure is well known. The problem of a membrane with no initial tension was first solved by Hencky (1). In the present paper Hencky's solution is generalized to include the case of an arbitrary initial tension, and curves are derived showing the transition between the results for the two extreme cases previously considered.

1. Introduction

THE deflexion of an initially tensioned circular membrane subjected to a uniform pressure may be easily calculated on the assumption that the tension does not change sensibly when the pressure is applied. Let a be the radius of the membrane, h its thickness, and σ the initial tensile stress. Then with the above assumption the deflexion w of points distant r from the centre due to a uniform pressure p is given by

$$w = \left(\frac{a}{4}\right)\left(\frac{pa}{\sigma h}\right)\left(1 - \frac{r^2}{a^2}\right).$$

The problem of a membrane with no initial tension is less simple, as the governing equation is non-linear. This case was first considered by Hencky (1) who showed that the deflexion is given by

$$w = a\left(\frac{pa}{Eh}\right)^{\frac{1}{3}} f_1(r),$$

where E is Young's modulus and $f_1(r)$ is a function of r which may be expressed as a power series in r^2.

Hencky also showed that the radial stress σ_r and the tangential stress σ_θ are given by

$$\sigma_r = \frac{E}{4}\left(\frac{pa}{Eh}\right)^{\frac{2}{3}} f_2(r)$$

and

$$\sigma_\theta = \frac{E}{4}\left(\frac{pa}{Eh}\right)^{\frac{2}{3}} f_3(r),$$

where $f_2(r)$ and $f_3(r)$ are functions of r expressible as power series in r^2.

It is evident that these two treatments are limiting cases of a more general theory in which the initial tension is finite and the variation of the tension with the applied pressure is taken into account.

In the present paper this theory is developed and curves are derived showing the transition between the two extreme cases considered above.

2. Theory

Let σ_r, σ_θ be the increments to the radial and tangential stresses respectively when the membrane is subjected to a uniform pressure p.

Then the equation of radial equilibrium is

$$\sigma_\theta = \frac{d}{dr}(r\sigma_r) \tag{1}$$

and the equation of normal equilibrium is

$$(\sigma + \sigma_r)\frac{dw}{dr} = -\frac{pr}{2h}, \tag{2}$$

where σ is the initial stress and w is the deflexion at radius r.

If u be the radial displacement due to the pressure, the incremental radial strain is given by

$$\epsilon_r = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \tag{3}$$

and the incremental tangential strain by

$$\epsilon_\theta = \frac{u}{r}. \tag{4}$$

The incremental stresses and strains are related by the equations

$$\sigma_r - \nu\sigma_\theta = E\epsilon_r \tag{5}$$

and

$$\sigma_\theta - \nu\sigma_r = E\epsilon_\theta, \tag{6}$$

where ν is Poisson's ratio.

Eliminating ϵ_r, ϵ_θ, and u from equations (3), (4), (5), and (6), we obtain the relation

$$r\frac{d}{dr}(\sigma_r + \sigma_\theta) + \frac{E}{2}\left(\frac{dw}{dr}\right)^2 = 0. \tag{7}$$

Eliminating dw/dr from equations (2) and (7) gives

$$\frac{1}{r}(\sigma + \sigma_r)^2\frac{d}{dr}(\sigma_r + \sigma_\theta) + \frac{Ep^2}{8h^2} = 0. \tag{8}$$

It is convenient to write equation (8) in the non-dimensional form

$$\frac{1}{\rho}\left(\frac{\sigma + \sigma_r}{k}\right)^2\frac{d}{d\rho}\left(\frac{\sigma_r + \sigma_\theta}{k}\right) + 8 = 0, \tag{9}$$

where

$$\rho \equiv r/a \quad \text{and} \quad k \equiv \frac{E}{4}\left(\frac{pa}{Eh}\right)^{\frac{2}{3}}.$$

From symmetry the stresses must be even functions of r and following Hencky we assume a power series for the total radial stress

$$\sigma+\sigma_r = k(B_0+B_2\rho^2+B_4\rho^4+...), \tag{10}$$

where the B's are non-dimensional.

Using equation (1) we obtain for the total tangential stress

$$\sigma+\sigma_\theta = k(B_0+3B_2\rho^2+5B_4\rho^4+...). \tag{11}$$

Substitution of (10) and (11) into (9) gives the equation

$$(B_0+B_2\rho^2+B_4\rho^4+...)^2(B_2+3B_4\rho^2+6B_6\rho^4+10B_8\rho^6+...)+1 = 0. \tag{12}$$

Expanding this and equating to zero coefficients of powers of ρ^2, we obtain

$$B_2 = -\frac{1}{B_0^2}, \qquad B_4 = -\frac{2}{3B_0^5},$$

$$B_6 = -\frac{13}{18B_0^8}, \qquad B_8 = -\frac{17}{18B_0^{11}},$$

$$B_{10} = -\frac{37}{27B_0^{14}}, \qquad B_{12} = -\frac{1205}{567B_0^{17}},\dagger \text{ etc.}$$

The constant B_0 is determined by the boundary condition at $r = a$. Assuming that the membrane is fixed to a rigid support at its edge, this condition is $u = 0$; hence from (4) and (6),

$$\sigma_\theta-\nu\sigma_r = 0 \quad \text{when } \rho = 1.$$

This gives

$$(1-\nu)B_0+(3-\nu)B_2+(5-\nu)B_4+... = (1-\nu)\sigma/k$$

or

$$B_0\left(1-\frac{3-\nu}{1-\nu}\frac{1}{B_0^3}-\frac{5-\nu}{1-\nu}\frac{2}{3B_0^6}-...\right) = \sigma/k. \tag{13}$$

From (10) and (11) the central incremental stress σ_0 is given by

$$\sigma_0 = kB_0-\sigma$$

and hence from (13)

$$\frac{\sigma_0}{k} = B_0\left(\frac{3-\nu}{1-\nu}\frac{1}{B_0^3}+\frac{5-\nu}{1-\nu}\frac{2}{3B_0^6}+...\right). \tag{14}$$

Taking Poisson's ratio ν as $0\cdot3$ in equations (13) and (14), values of σ/k and σ_0/k have been calculated for different values of B_0, and σ_0/k is shown as a function of σ/k in Fig. 1.

It is found that when $\sigma = 0$, i.e. there is no initial tension, the value of B_0 is $1\cdot724\ddagger$ and hence

$$\frac{\sigma_0}{k} = 1\cdot724$$

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† Hencky (loc. cit.) gives the erroneous value $-407/189B_0^{17}$.

‡ Owing to errors in the evaluation of his equation (11 a), Hencky (loc. cit.) obtains the value $1\cdot713$.

or
$$\sigma_0 = 0{\cdot}431\,E\!\left(\frac{pa}{Eh}\right)^{\!\frac{2}{3}}. \tag{15}$$
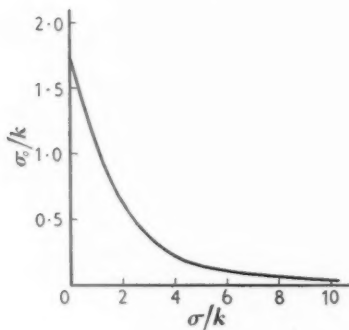

Fig. 1.

For large values of σ, B_0 becomes large and we may write

$$B_0 \simeq \frac{\sigma}{k}$$

and hence from (14)
$$\frac{\sigma_0}{k} \simeq \frac{3-\nu}{1-\nu}\!\left(\frac{\sigma}{k}\right)^{\!-1}$$

or
$$\sigma_0 \simeq \frac{3-\nu}{1-\nu}\frac{E}{64}\!\left(\frac{pa}{\sigma h}\right)^{\!2} = 0{\cdot}0603\,E\!\left(\frac{pa}{\sigma h}\right)^{\!2}. \tag{16}$$

It thus appears that whereas for an untensioned membrane the incremental stress is proportional to the $\frac{2}{3}$ power of the applied pressure, that for a highly tensioned membrane is proportional to the square of the applied pressure. In order to illustrate the transition between these two laws, it is convenient to use the non-dimensional variables

$$P \equiv \frac{pa}{Eh}\!\left(\frac{\sigma}{E}\right)^{\!-\frac{3}{2}}, \qquad S_0 \equiv \sigma_0/\sigma$$

to represent the pressure and the incremental centre stress respectively. Using the definition of k, we may write

$$P = 8\!\left(\frac{\sigma}{k}\right)^{\!-\frac{3}{2}}, \qquad S_0 = \frac{\sigma_0}{k}\!\left(\frac{\sigma}{k}\right)^{\!-1}.$$

The curve of Fig. 1 has been used to obtain the relation between P and S_0, which is plotted linearly in Fig. 2 and logarithmically in Fig. 3.

In terms of P and S_0, equations (15) and (16) become respectively

$$S_0 = 0.431 P^{\frac{2}{3}} \tag{17}$$
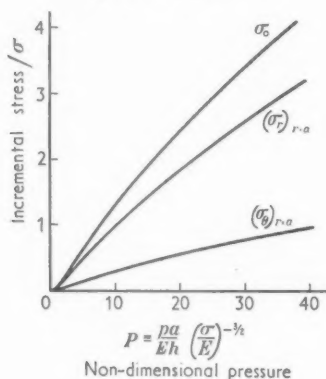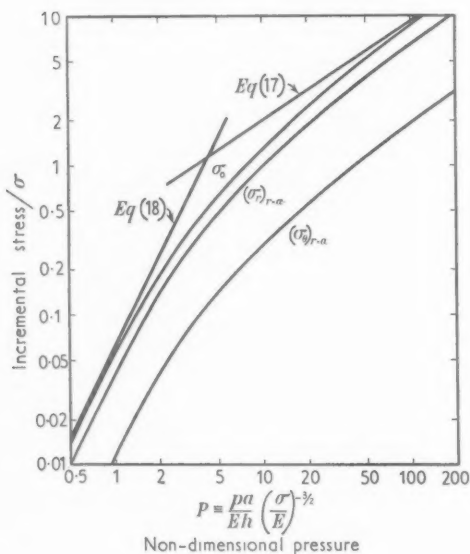
and

$$S_0 = 0.0603 P^2. \tag{18}$$

FIG. 2.

FIG. 3.

The asymptotes corresponding to these equations are shown in Fig. 3. It appears that for $P > 100$ equation (17) gives S_0 within about 10 per cent., and for $P < 1$ equation (18) gives S_0 to the same accuracy.

It may be seen from Fig. 2 that for a certain range of pressure the incremental stresses (and hence the incremental strains) are approximately proportional to the applied pressure. Thus for $5 < P < 15$ the central stress may be represented by the equation

$$\sigma_0 = 0.129 \frac{pa}{h} \left(\frac{\sigma}{E}\right)^{-\frac{1}{2}}$$

to an accuracy of ± 2 per cent., and for $0 < P < 20$ by the equation

$$\sigma_0 = 0.125 \frac{pa}{h} \left(\frac{\sigma}{E}\right)^{-\frac{1}{2}}$$

within ± 3 per cent. of its value at $P = 20$.

The curves giving the incremental radial and tangential stresses at $r = a$ (Figs. 2 and 3) have been calculated from equations (10) and (11). These stresses are very nearly proportional to σ_0 for all values of P and may be represented within about 2 per cent. by the equations

$$(\sigma_r)_{r=a} = 0.756\sigma_0$$

and

$$(\sigma_\theta)_{r=a} = 0.227\sigma_0.$$

The dependence of σ_r and σ_θ on r is given by equations (10) and (11); when σ is large the relations are parabolic. Fig. 4 shows the curves for the extreme cases $\sigma = 0$ and $\sigma = \infty$.

From equations (5) and (6) we have

$$\frac{\epsilon_r}{\epsilon_0} = \frac{\sigma_r/\sigma_0 - \nu\sigma_\theta/\sigma_0}{1-\nu} \tag{19}$$

and

$$\frac{\epsilon_\theta}{\epsilon_0} = \frac{\sigma_\theta/\sigma_0 - \nu\sigma_r/\sigma_0}{1-\nu}, \tag{20}$$

where $\epsilon_0 \equiv (1-\nu)\sigma_0/E$, the incremental strain at $r = 0$. The strain distribution given by equations (19) and (20) is shown in Fig. 5 for the extreme cases $\sigma = 0$ and $\sigma = \infty$. It is seen from Figs. 4 and 5 that the distribution of incremental stress and strain varies but little with the initial stress σ.

The deflexion of the membrane is governed by equation (2), which may be written in the non-dimensional form

$$\frac{\sigma+\sigma_r}{k} \frac{d(w/c)}{d\rho} = -2\rho, \tag{21}$$

where

$$c \equiv a\left(\frac{pa}{Eh}\right)^{\frac{1}{3}}.$$

Assume that w is given by an even power series

$$w = c(A_0 + A_2\rho^2 + A_4\rho^4 + ...), \tag{22}$$

where the coefficients A are non-dimensional. Substitution of (10) and (22) into (21) then gives the equation

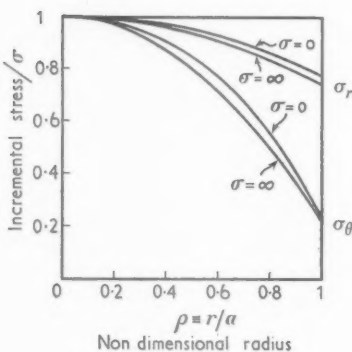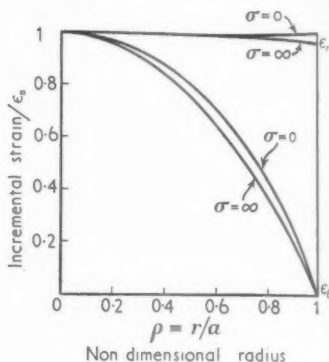$$(B_0 + B_2\rho^2 + B_4\rho^4 + ...)(A_2 + 2A_4\rho^2 + 3A_6\rho^4 + ...) + 1 = 0. \quad (23)$$

FIG. 4.

FIG. 5.

From this we obtain, after substituting for B_2, B_4,... in terms of B_0,

$$A_2 = -\frac{1}{B_0}, \qquad A_4 = -\frac{1}{2B_0^4},$$

$$A_6 = -\frac{5}{9B_0^7}, \qquad A_8 = -\frac{55}{72B_0^{10}},$$

$$A_{10} = -\frac{7}{6B_0^{13}}, \qquad A_{12} = -\frac{205}{108B_0^{16}}, \text{ etc.}$$

The constant A_0 is determined by the condition that the deflexion w is zero when $r = a$; hence from (22)

$$A_0 = -(A_2 + A_4 + A_6 + ...)$$

or $$A_0 = \frac{1}{B_0}\left(1 + \frac{1}{2B_0^3} + \frac{5}{9B_0^6} + \frac{55}{72B_0^9} + \frac{7}{6B_0^{12}} + \frac{205}{108B_0^{15}} + ...\right). \quad (24)$$

Elimination of B_0 from equations (13) and (24) determines A_0 as a function of σ/k. From (22) the central deflexion $w_0 = cA_0$ and hence w_0/c is known as a function of σ/k. This function is plotted in Fig. 6. When $\sigma = 0$, $w_0/c = 0.653$ and hence

$$w_0 = 0.653a\left(\frac{pa}{Eh}\right)^{\frac{1}{3}}. \quad (25)$$

For large values of σ, B_0 is large and approximately equal to σ/k, so that we may write

$$A_0 \simeq \left(\frac{\sigma}{k}\right)^{-1}$$

and hence
$$w_0 \simeq c\left(\frac{\sigma}{k}\right)^{-1} = \frac{a}{4}\left(\frac{pa}{\sigma h}\right). \qquad (26)$$

This is the well-known result for the central deflexion of a tightly stretched membrane subjected to uniform pressure.
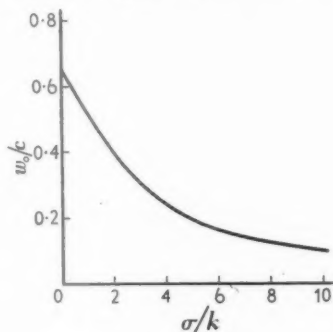

FIG. 6.

In order to illustrate the transition from the $\frac{1}{3}$ power law of equation (25) to the linear relationship of equation (26), it is convenient to use the non-dimensional variable
$$W_0 \equiv \frac{w_0}{a}\left(\frac{\sigma}{E}\right)^{-\frac{1}{2}},$$

together with P as previously defined. Then we may write as before
$$P = 8\left(\frac{\sigma}{k}\right)^{-\frac{3}{2}}$$

and
$$W_0 = 2\left(\frac{w_0}{c}\right)\left(\frac{\sigma}{k}\right)^{-\frac{1}{2}}.$$

In terms of P and W_0, equations (25) and (26) become respectively
$$W_0 = 0.653 P^{\frac{1}{3}} \qquad (27)$$

and
$$W_0 = 0.25 P. \qquad (28)$$

The curve of Fig. 6 has been used to obtain the relation between P and W_0, which is plotted linearly in Fig. 7 and logarithmically in Fig. 8. In Figs. 7 and 8 the asymptotes corresponding to equations (27) and (28) are also shown. It appears that for $P > 100$ equation (27) gives W_0 within about 5 per cent., and for $P < 1$ equation (28) gives W_0 to the same accuracy. Thus the expression (26) for the central deflexion of a stretched membrane is inaccurate when
$$\frac{pa}{Eh} > \left(\frac{\sigma}{E}\right)^{\frac{3}{2}}.$$

Eq.(27)

Eq.(28)

Non-dimensional central deflection $W_0 \equiv \dfrac{w_0}{a}\left(\dfrac{\sigma}{E}\right)^{-1/2}$

$P \equiv \dfrac{pa}{Eh}\left(\dfrac{\sigma}{E}\right)^{-3/2}$

Non-dimensional pressure

Fig. 7.

Eq.(27)

Eq.(28)

Non dimensional central deflection $W_0 \equiv \dfrac{w_0}{a}\left(\dfrac{\sigma}{E}\right)^{-1/2}$

$P \equiv \dfrac{pa}{Eh}\left(\dfrac{\sigma}{E}\right)^{-3/2}$

Non-dimensional pressure

Fig. 8.

$\sigma = 0$

$\sigma = \infty$

Deflection/w_0

$\rho \equiv r/a$

Non-dimensional radius

Fig. 9.

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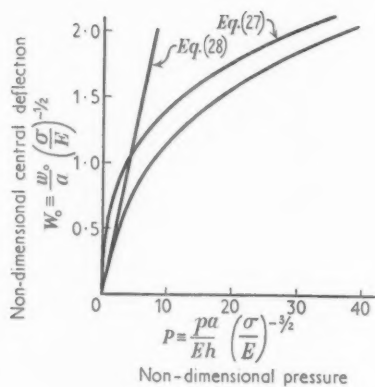3. Co
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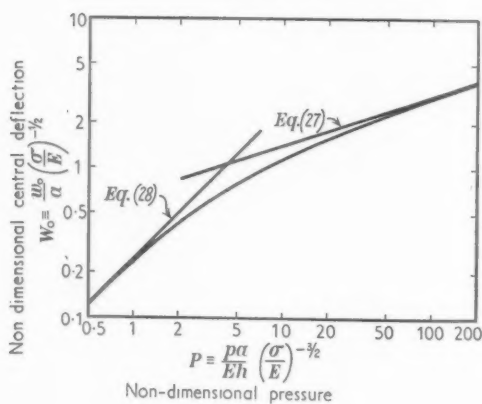
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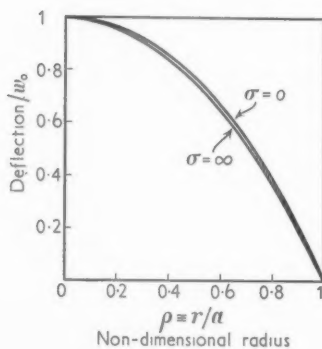cent.,
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The dependence of w on r is given by equation (22); for large values of σ, the relation is parabolic. Fig. 9 shows the deflexion curves for the extreme cases $\sigma = 0$ and $\sigma = \infty$, and it is seen that the difference between them is small.

3. Conclusions

(a) For large initial tension or small applied pressure such that

$$\frac{pa}{Eh} < \left(\frac{\sigma}{E}\right)^{\frac{3}{2}},$$

equation (16) gives the incremental stress at the centre within about 10 per cent., and equation (26) gives the central deflexion within about 5 per cent.

(b) For small initial tension or large applied pressure such that

$$\frac{pa}{Eh} > 100\left(\frac{\sigma}{E}\right)^{\frac{3}{2}},$$

the Hencky formulae for the incremental stresses hold within about 10 per cent., and that for the deflexion holds within about 5 per cent.

(c) The incremental stresses and strains are approximately proportional to the applied pressure for pressures between

$$5\frac{Eh}{a}\left(\frac{\sigma}{E}\right)^{\frac{3}{2}} \quad \text{and} \quad 15\frac{Eh}{a}\left(\frac{\sigma}{E}\right)^{\frac{3}{2}}.$$

(d) The distribution of incremental stress and strain in the membrane is almost independent of the initial tension.

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ON THE VIBRATION OF A CANTILEVER PLATE†

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SUMMARY

By following a procedure similar to the Rayleigh–Ritz method, approximate equations are obtained for the frequencies of vibration of a cantilever plate. Some of the results are compared with experiment.

1. THE partial differential equation (1)

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} = \lambda w, \tag{1.1}$$

where (x, y) lies in the rectangle $0 \leqslant x \leqslant l,\ -\frac{1}{2}b \leqslant y \leqslant \frac{1}{2}b$, together with the boundary conditions

$$\frac{\partial^2 w}{\partial x^2} + \sigma\frac{\partial^2 w}{\partial y^2} = \frac{\partial^3 w}{\partial x^3} + (2-\sigma)\frac{\partial^3 w}{\partial x \partial y^2} = 0, \quad \text{on } x = l,$$

$$\frac{\partial^2 w}{\partial y^2} + \sigma\frac{\partial^2 w}{\partial x^2} = \frac{\partial^3 w}{\partial y^3} + (2-\sigma)\frac{\partial^3 w}{\partial x^2 \partial y} = 0, \quad \text{on } y = \pm\tfrac{1}{2}b, \tag{1.2}$$

and

$$w = \frac{\partial w}{\partial x} = 0, \quad \text{on } x = 0,$$

are the equations for the normal modes of transverse vibration of a thin rectangular plate, built in at the end $x = 0$. Here l and b denote the length and breadth of the plate, σ is Poisson's ratio, and $w = w(x, y)$ is the deflexion of the plate, supposed small, in a direction perpendicular to its plane of equilibrium. The parameter λ is given in terms of the angular frequency p by the formula $\lambda = 12\rho(1-\sigma^2)p^2/Eh^2$, where ρ is the density, E is Young's modulus, and h is the thickness of the plate.

The Rayleigh–Ritz approximate method (2, 3) for the solution of (1.1) and (1.2) consists in minimizing the ratio between the strain and kinetic energies of the plate; the minimum yields the natural frequency of vibration. The Rayleigh–Ritz principle works well for the determination of the fundamental frequency, but, for overtones, the work becomes laborious.‡ A method will be developed in this paper which yields relatively simple approximate formulae, and which enables calculations for overtone frequencies to be made fairly easily.

† This work was carried out at the Bristol Aeroplane Company.

‡ Any approximating mode function for an overtone must be made orthogonal to the fundamental and any lower overtone.

According to the calculus of variations, the partial differential equation (1.1) and the first two equations in the boundary conditions (1.2) may be obtained by carrying out the variation (4)

$$\delta \int_0^l \int_{-\frac{1}{2}b}^{\frac{1}{2}b} \left[\left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\sigma)\left(\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right) \right\} - \lambda w^2 \right] dx dy = 0 \tag{1.3}$$

subject to the condition that w satisfies the last equation in (1.2).

Following the procedure adopted by Rayleigh and Ritz, in attempting to determine numerical values of the overtone frequencies of a cantilever plate, it is natural to substitute a single product†

$$w = u(x)v(y) \tag{1.4}$$

in (1.3). For the case when the edges parallel to the x-axis are free, we may take $v(y)$ equal to one of the functions representing the normal modes of vibration of the free-free beam. In carrying out the variation (1.3), we shall regard $u(x)$ as the independent variable function, subject only to the condition that

$$u = u' = 0, \quad \text{on } x = 0, \tag{1.5}$$

where $u' = du/dx$. This procedure leads to an ordinary differential equation for the determination of u; and the frequencies of vibration, corresponding to modes of the type (1.4), may then be calculated. The results are found to agree quite well with experimental values.

2. Several writers have used a convenient notation for the classification of modes (5, 6). In the case of a cantilever plate, a mode of vibration is denoted by the pair of whole numbers m, n if there are m nodal lines running 'parallel' to the x-axis and n nodal lines running 'parallel' to the y-axis, the x, y-axes being those defined at the beginning of section 1. From considerations of symmetry with respect to the centre line of the plate, it follows that for pure flexure m is even, whilst for pure torsion m is odd. Grinsted's paper gives several illustrations of different m, n modes of vibration of the cantilever plate. It is noteworthy that (1.4) corresponds to the m, n mode of vibration, where m is the number of zeros of $v(y)$ in $-\frac{1}{2}b < y < \frac{1}{2}b$ and n the number of zeros of $u(x)$ in $0 < x < l$. Pure flexure and pure torsion will be given, respectively, by the even and odd modes of vibration of the free-free beam.

† As in the Rayleigh–Ritz method, (1.4) is to be regarded as an approximation to the actual mode of vibration.

3. Substituting (1.4) in (1.3), we have, for any given $v(y)$,

$$I_1\delta\int_0^l (u'')^2\,dx +2(1-\sigma)I_2\delta\int_0^l (u')^2\,dx +2\sigma I_3\delta\int_0^l u''u\,dx +$$

$$+(I_4-\lambda I_1)\delta\int_0^l u^2\,dx = 0, \qquad (3.1)$$

where

$$I_1 = \int_{-\frac12 b}^{\frac12 b} v^2\,dy, \qquad I_2 = \int_{-\frac12 b}^{\frac12 b} (v')^2\,dy, \qquad I_3 = \int_{-\frac12 b}^{\frac12 b} v''v\,dy, \qquad I_4 = \int_{-\frac12 b}^{\frac12 b} (v'')^2\,dy,$$

and

$$v' = dv/dy, \qquad v'' = d^2v/dy^2.$$

Let $J_2 = b^2 I_2/I_1$, $J_3 = b^2 I_3/I_1$, $J_4 = b^4 I_4/I_1$, and $\omega^4 = b^4\lambda$. Then, by carrying out the variations in (3.1) subject to the boundary conditions (1.5), we obtain the differential equation

$$\frac{d^4u}{dx^4} - \frac{2}{b^2}\{J_2-\sigma(J_2+J_3)\}\frac{d^2u}{dx^2} - \frac{\omega^4-J_4}{b^4}u = 0, \qquad (3.2)$$

together with the boundary conditions

$$\frac{d^2u}{dx^2} + \frac{\sigma}{b^2}J_3 u = \frac{d^3u}{dx^3} - \frac{2}{b^2}\{J_2-\sigma(J_2+\tfrac12 J_3)\}\frac{du}{dx} = 0, \quad \text{on } x = l. \qquad (3.3)$$

The differential system given by (1.5), (3.2), and (3.3) may be solved in a convenient manner. As fundamental solutions of (3.2), we have

$$\begin{matrix}\sinh\\\cosh\end{matrix}(\alpha x/b) \quad \text{and} \quad \begin{matrix}\sin\\\cos\end{matrix}(\beta x/b),$$

where

$$\alpha^2 = \sqrt{\{\omega^4-J_4+[J_2-\sigma(J_2+J_3)]^2\}} + [J_2-\sigma(J_2+J_3)], \qquad (3.4)$$

and

$$\beta^2 = \sqrt{\{\omega^4-J_4+[J_2-\sigma(J_2+J_3)]^2\}} - [J_2-\sigma(J_2+J_3)].$$

Thus the solution of (3.2) satisfying (1.5) is

$$u = A[\cosh(\alpha x/b)-\cos(\beta x/b)] + B\left[\frac{1}{\alpha}\sinh(\alpha x/b) - \frac{1}{\beta}\sin(\beta x/b)\right]. \qquad (3.5)$$

The constants A and B are obtained by substituting (3.5) in (3.3). We obtain a pair of homogeneous equations from which A and B may be eliminated to give an equation for ω. By using (3.4), this equation may be expressed in the form

$$\omega^4-J_4-2\sigma J_3\{J_2-\sigma(J_2+\tfrac12 J_3)\} +$$

$$+[\omega^4-J_4+(1-\sigma)^2 J_2^2+\{J_2-\sigma(J_2+J_3)\}^2]\cosh(\alpha l/b)\cos(\beta l/b) +$$

$$+\left[\{J_2-\sigma(J_2-J_3)\}\sqrt{(\omega^4-J_4)} - \sigma^2 J_3^2\frac{\{J_2-\sigma(J_2+J_3)\}}{\sqrt{(\omega^4-J_4)}}\right]\sinh(\alpha l/b)\sin(\beta l/b) = 0.$$

$$(3.6)$$

Equation (3.6) gives the required frequencies; its solution will be obtained in a subsequent section.

If we carry out the variations in (3.1) subject to the single boundary condition $u = 0$, on $x = 0$ (this is the case of the simply supported 'cantilever' plate), then (3.2) and (3.3) result as before; however, (1.5) is to be replaced by

$$u = u'' = 0, \quad \text{on } x = 0.$$

In this case we obtain $u = A\sinh(\alpha x/b) + B\sin(\beta x/b)$; and the equation for ω may be expressed in the form

$$\frac{(\beta^2 - \sigma J_3)^2}{\beta}\tan(\beta l/b) = \frac{(\alpha^2 + \sigma J_3)^2}{\alpha}\tanh(\alpha l/b). \tag{3.7}$$

In the work considered here, we are taking $u(x)$ to be the independent variable function and $v(y)$ to be initially given. We could, of course, carry out the work with the parts played by u and v interchanged (e.g. we might take u to be a given function representing a mode of vibration of the fixed-free beam); v would then satisfy equations of the same type as (3.2) and (3.3). A more difficult enterprise would be to take both u and v as two independent variable functions; we should then obtain two sets of differential systems of the type (3.2) and (3.3), each set involving integrals of the function in the other set.

4. Calculation of the integrals I_1, I_2, I_3, and I_4

The functions representing the modes of vibration of a free-free beam are given by

$$v = K_1(y) + K_2(y), \qquad -\tfrac{1}{2}b \leqslant y \leqslant \tfrac{1}{2}b,$$

where

$$K_1(y) = \cos s(y + \tfrac{1}{2}b) + \mu\sin s(y + \tfrac{1}{2}b),$$

$$K_2(y) = \cosh s(y + \tfrac{1}{2}b) + \mu\sinh s(y + \tfrac{1}{2}b),$$

and

$$\mu = (\cos sb - \cosh sb)/(\sinh sb - \sin sb),$$

and s runs through the positive roots of the equation

$$\cos sb \cosh sb = 1. \tag{4.1}$$

In the notation of section 2, these functions correspond to $m = 2, 3, 4, \dots$. For $m = 0$ and 1, we take $v = 1$ and y, respectively; these functions also satisfy the differential equation and the boundary conditions of a free-free beam.

Since $K_1'' = -s^2 K_1$, $K_2'' = s^2 K_2$, $K_1(\pm\tfrac{1}{2}b) = K_2(\pm\tfrac{1}{2}b)$, and

$$K_1'(\pm\tfrac{1}{2}b) = K_2'(\pm\tfrac{1}{2}b),$$

we have

$$\int_{-\frac{1}{2}b}^{\frac{1}{2}b} K_1(y)K_2(y)\,dy = \frac{1}{2s^2}\int_{-\frac{1}{2}b}^{\frac{1}{2}b}(K_1 K_2'' - K_2 K_1'')\,dy$$

$$= \frac{1}{2s^2}[K_1 K_2' - K_2 K_1']_{-\frac{1}{2}b}^{\frac{1}{2}b}$$

$$= 0. \tag{4.2}$$

Let $\eta = sb$ and $t = \tan\frac{1}{2}\eta$; by (4.1), we obtain

$$\sin\eta = \frac{2t}{1+t^2}, \quad \cos\eta = \frac{1-t^2}{1+t^2}, \quad \cosh\eta = \frac{1+t^2}{1-t^2}, \quad \sinh\eta = \mp\frac{2t}{1-t^2}.$$

In the last equation the upper sign holds if $m = 2, 4, 6,\ldots$ whilst the lower sign holds if $m = 3, 5, 7,\ldots$. For the first case it is seen that $\mu = t$, and for the second case $\mu = -1/t$.

By direct integration, we obtain

$$\int_{-\frac{1}{2}b}^{\frac{1}{2}b} K_1(y)^2\,dy = \begin{cases} \frac{1}{2}b\left(1+t^2+\dfrac{2t}{\eta}\right), & m = 2, 4, 6,\ldots \\[2mm] \frac{1}{2}b\left(1+\dfrac{1}{t^2}-\dfrac{2}{\eta t}\right), & m = 3, 5, 7,\ldots \end{cases}$$

and

$$\int_{-\frac{1}{2}b}^{\frac{1}{2}b} K_2(y)^2\,dy = \begin{cases} \frac{1}{2}b\left(1-t^2-\dfrac{2t}{\eta}\right), & m = 2, 4, 6,\ldots \\[2mm] \frac{1}{2}b\left(1-\dfrac{1}{t^2}+\dfrac{2}{\eta t}\right), & m = 3, 5, 7,\ldots. \end{cases} \tag{4.3}$$

Hence, by (4.2) and (4.3),

$$I_1 = \int_{-\frac{1}{2}b}^{\frac{1}{2}b} v^2\,dy = \int_{-\frac{1}{2}b}^{\frac{1}{2}b} K_1^2\,dy + 2\int_{-\frac{1}{2}b}^{\frac{1}{2}b} K_1 K_2\,dy + \int_{-\frac{1}{2}b}^{\frac{1}{2}b} K_2^2\,dy = b.$$

Next, since $v'' = K_1'' + K_2'' = -s^2(K_1 - K_2)$,

$$I_3 = \int_{-\frac{1}{2}b}^{\frac{1}{2}b} v''v\,dy = -s^2\int_{-\frac{1}{2}b}^{\frac{1}{2}b}(K_1^2 - K_2^2)\,dy$$

$$= \begin{cases} -\dfrac{\eta^2}{b}\left(t^2+\dfrac{2t}{\eta}\right), & m = 2, 4, 6,\ldots \\[2mm] -\dfrac{\eta^2}{b}\left(\dfrac{1}{t^2}-\dfrac{2}{\eta t}\right), & m = 3, 5, 7,\ldots. \end{cases}$$

Also,

$$I_4 = \int_{-\frac{1}{2}b}^{\frac{1}{2}b} (v'')^2 \, dy = s^4 \left\{ \int_{-\frac{1}{2}b}^{\frac{1}{2}b} K_1^2 \, dy - 2 \int_{-\frac{1}{2}b}^{\frac{1}{2}b} K_1 K_2 \, dy + \int_{-\frac{1}{2}b}^{\frac{1}{2}b} K_2^2 \, dy \right\} = \frac{\eta^4}{b^3}.$$

Finally, since $K_1(\pm\frac{1}{2}b) = K_2(\pm\frac{1}{2}b)$ and $K_1'(\pm\frac{1}{2}b) = K_2'(\pm\frac{1}{2}b)$,

$$I_2 = \int_{-\frac{1}{2}b}^{\frac{1}{2}b} (v')^2 \, dy = [v'v]_{-\frac{1}{2}b}^{\frac{1}{2}b} - \int_{-\frac{1}{2}b}^{\frac{1}{2}b} v''v \, dy$$

$$= 4[K_1' K_1]_{-\frac{1}{2}b}^{\frac{1}{2}b} - I_3$$

$$= \begin{cases} \dfrac{\eta^2}{b}\left(t^2 - \dfrac{6t}{\eta}\right), & m = 2, 4, 6,\dots \\[2mm] \dfrac{\eta^2}{b}\left(\dfrac{1}{t^2} + \dfrac{6}{\eta t}\right), & m = 3, 5, 7,\dots; \end{cases}$$

where, in the last two lines, we have used the equation defining K_1.

For the coefficients J_2, J_3, and J_4, we have

$$J_2 = \eta^2\left(t - \frac{6t}{\eta}\right), \qquad J_3 = -\eta^2\left(t^2 + \frac{2t}{\eta}\right), \qquad J_4 = \eta^4,$$

for $m = 2, 4, 6,\dots$. If $m = 3, 5, 7,\dots$, we replace t by $-1/t$ in these formulae. For $m = 0$, we have $J_2 = J_3 = J_4 = 0$. For $m = 1$, we have $J_2 = 12$, $J_3 = J_4 = 0$. Table 1 gives the approximate numerical values.[†]

<div align="center">TABLE 1</div>

m	0	1	2	3	4	5	6
η	—	—	4·73	7·85	11	14·14	17·28
t	—	—	−0·982	+1	−1	+1	−1
J_2	0	12	49·4	108·7	185·6	284·5	399·5
J_3	0	0	−12·3	−46	−98	−171·8	−261·6
J_4	0	0	500·5	3803	14618	39970	89120

5. Method of solution of equation (3.6)

If $m = 0$, then, by equating J_2, J_3, and J_4 to zero in (3.6), we obtain the well-known equation corresponding to the vibration of the cantilever beam. Let $f_{m,n}$ be the frequency corresponding to the m, n mode of vibration of the cantilever plate. It follows that the frequencies $f_{0,n}$, $n = 0, 1, 2, 3,\dots$,

† The values of η are taken from Timoshenko's book *Vibration Problems in Engineering*, p. 343.

are given by the usual formula for the frequency of vibration of a cantilever beam, apart from a factor $(1-\sigma^2)$. We have

$$f_{0,n} = \frac{k_n^2}{2\pi} \sqrt{\left/\left(\frac{E}{12\rho(1-\sigma^2)}\right)\right.} \frac{h}{l^2}, \tag{5.1}$$

where k_0, k_1,... are the roots of $\cosh k \cos k = -1$.

It may be anticipated that $\omega^4 > J_4 \; (= \eta^4)$; and hence, if $n \geqslant 1$, that $\alpha > \sqrt{\{24(1-\sigma)\}}$. Thus, if $\sqrt{(1-\sigma)}l/b > 1$, we may make the approximations $\tanh \alpha l/b = 1$ and $\operatorname{sech} \alpha l/b = 0$ in (3.6). Writing

$$L = \omega^4 - J_4 + (1-\sigma)^2 J_2^2 + \{J_2 - \sigma(J_2+J_3)\}^2,$$

$$M = \{J_2 - \sigma(J_2-J_3)\}\sqrt{(\omega^4-J_4)} - \sigma^2 J_3^2 \frac{\{J_2-\sigma(J_2+J_3)\}}{\sqrt{(\omega^4-J_4)}}, \tag{5.2}$$

dividing (3.6) by $\cosh(\alpha l/b)$, and making the above approximations, we obtain $\tan(\beta l/b) = -L/M$; or

$$\frac{l}{b} = \frac{(n+1)\pi - \tan^{-1}(L/M)}{\beta}. \tag{5.3}$$

In this equation n is zero or a positive integer and is to be identified with the n defined in section 2. The branch of the inverse tangent is to be that which lies in $(0, \pi)$. The right-hand side of (5.3) is independent of l and b. For a given η (i.e. given m), its graph may be plotted against values of ω for each $n = 0, 1, 2,...$ (Fig. 1). Before this is done it is necessary to assume an appropriate value for σ; thus different graphs must be plotted for different materials. In the case of ordinary torsion, given by $m = 1$, we have $J_3 = J_4 = 0$, and (5.3) may be expressed in a form for which the same graph holds for all materials. Thus, writing

$$\xi = \omega^2/(1-\sigma)J_2 = \omega^2/12(1-\sigma),$$

so that $L/M = \xi+2/\xi$ and $\beta = \sqrt{\{12(1-\sigma)\}}\{\sqrt{(\xi^2+1)}-1\}^{\frac{1}{2}}$, (5.3) is equal to

$$\sqrt{\{12(1-\sigma)\}}\frac{l}{b} = \frac{(n+1)\pi - \tan^{-1}(\xi+2/\xi)}{\{\sqrt{(\xi^2+1)}-1\}^{\frac{1}{2}}}. \tag{5.4}$$

The right-hand side of this equation is independent of σ; its graph may be plotted for various values of ξ and each $n = 0, 1, 2,...$ (Fig. 2).

Having plotted the graph appropriate to a particular m, n pair, we may obtain the solution ω of (5.3), or the solution ξ of (5.4), for a given l/b. In the case of (5.3), the frequency is given by

$$f_{m,n} = \frac{1}{2\pi}\left(\frac{E}{12\rho(1-\sigma^2)}\right)^{\frac{1}{2}}\frac{h}{b^2}\omega^2. \tag{5.5}$$
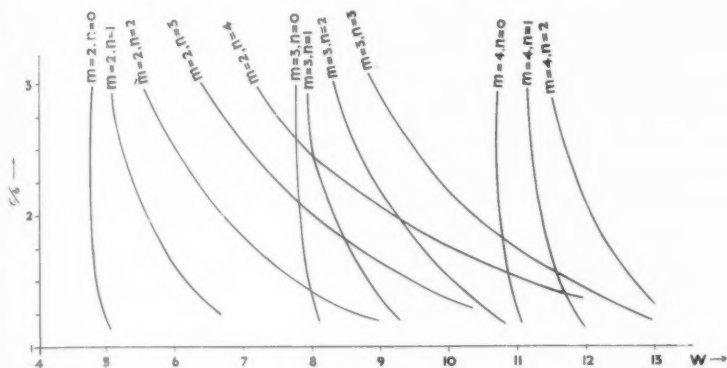
FIG. 1
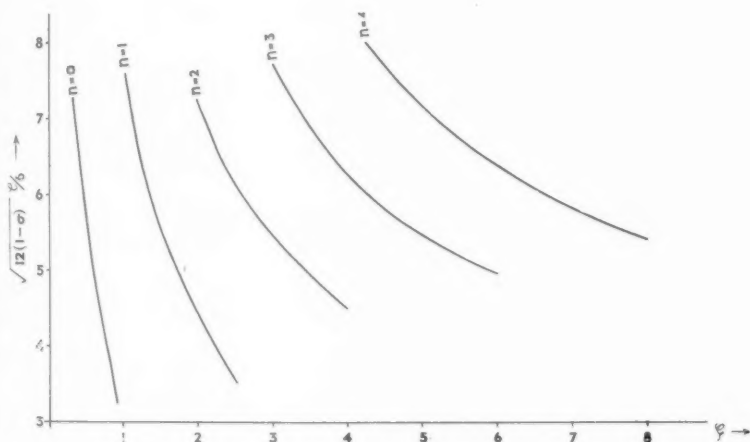
FIG. 2

In the case of (5.4), we have

$$f_{1,n} = \frac{\sqrt{3}}{\pi}\left(\frac{1-\sigma}{1+\sigma}\cdot\frac{E}{\rho}\right)^{\frac{1}{2}}\frac{h}{b^2}\xi. \qquad (5.6)$$

By following a similar procedure, an even more convenient method of solution can be discussed for the equation (3.7).

6. Comparison of results with experiment

In order to obtain some idea of the accuracy of the results, it is necessary to plot the appropriate graphs for the equations (5.3) and (5.4). Rough sketches of these graphs are shown in Figs. 1 and 2; in the case of

(5.3), σ was taken equal to 0·3, this being the approximate value for mild steel. For a specimen with $l = 5\cdot12''$, $b = 2\cdot76''$, and $h = 0\cdot053''$, Grinsted (6) gives several experimental values for the frequencies; some of these results are reproduced in Table 2. In this table, the upper integer, in the compartment corresponding to m, n, denotes the approximate theoretical value for the frequency, the lower integer is the experimental value, and the number immediately to the right of these two values represents the approximate percentage difference.

TABLE 2

Comparison of theoretical estimates for some frequencies of vibration of a cantilever plate with corresponding experimental values

	$n \to$ ▣	1	2	3	4	5
▣	69·5 / 64, 8·6%	436 / 405, 7·7%	1,220 / 1,120, 8·9%	2,390 / 2,233, 7%	3,940 / 3,736, 5·5%	5,900 / 5,573, 5·9%
1	276 / 260, 6·2%	905 / —,	1,743 / 1,676, 4%	2,970 / 2,804, 5·9%	4,530 / 4,335, 4·5%	
m 2 $\downarrow$	1,610 / 1,606, 0·2%	2,260 / —,	3,280 / 3,160, 3·8%	4,660 / 4,428, 5·3%	6,350 / 6,009, 5·7%	8,350 / 7,859, 5·9%
3	4,250 / 4,235, 0·4%	4,810 / 4,773, 0·8%	5,950 / 5,739, 3·7%	7,450 / 7,069, 5·4%	9,200 / —	11,280 / —
4	8,260 / 8,238, 0·3%	8,870 / 8,685, 2·1%	9,750 / 9,651, 1%	10,620 / —	13,150 / —	15,300 / —

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A SOLUTION OF TRANTER'S DUAL INTEGRAL EQUATIONS PROBLEM

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SUMMARY

Tranter (**1, 2**) has given the solution of the pair of integral equations (1) and (2) below in the form of a Neumann series of Bessel functions whose coefficients are determined by means of an infinite series of linear equations. This paper gives the formal solution as an integral, containing an unknown function which is determined by means of an integral equation of Fredholm's type. This solution appears to have both theoretical and practical advantages over Tranter's solution in that it is (in a sense) in closed form; moreover it does not depend on certain parameters being specified as large or small in order to complete a practical solution. Applications are given to several potential problems involving circular disks, some of these problems being believed to be new.

1. Statement of the problem

WE require to find a function $g(u)$ where

$$\int_0^\infty G(u)g(u)J_\nu(\rho u)\,du = h(\rho) \quad (0 < \rho < 1), \tag{1}$$

$$\int_0^\infty g(u)J_\nu(\rho u)\,du = 0 \quad (\rho > 1), \tag{2}$$

and where $G(u)$ is a known function of u.

2. The solution

We put

$$g(u) = \frac{2^k}{\Gamma(1-k)} u^{1+k} \int_0^1 f(t)t^{\alpha+1}J_{\nu-k}(ut)\,dt, \tag{3}$$

where $0 < \mathrm{re}(k) < 1$, and $f(t)$ is a function to be determined.

We write
$$A = 2/\Gamma(k)\Gamma(1-k), \tag{4}$$

so that the coefficient in equation (3) is $A2^{k-1}\Gamma(k)$. Here α may have any convenient value so long as the integral in equation (3) converges. It is not strictly necessary to insert the term $t^{\alpha+1}$ at all but it is convenient. The choice of k will be considered later.

The analysis will be purely formal. We shall show that the form of $g(u)$ in equation (3) satisfies equation (2) as it stands and will satisfy equation (1) if $f(t)$ satisfies a certain integral equation. There must be conditions

on $G(u)$, $f(t)$, and $h(\rho)$ in order that the integrals converge and that the order of integration may be inverted in what follows. We shall assume that the functions are such that the inversions of order made are legitimate.

3. Proof that equation (2) is satisfied

We substitute the value of $g(u)$ in the left-hand side of equation (2), whence, omitting a constant, we must show that

$$\int_0^\infty u^{1+k} J_\nu(\rho u)\, du \int_0^1 f(t) t^{\alpha+1} J_{\nu-k}(ut)\, dt = 0.$$

Here we may not invert the order of integration, but the left-hand side is

$$\int_0^\infty u^k J_\nu(\rho u)\, du \int_0^1 f(t) t^{\alpha-\nu+k} d\{t^{\nu-k+1} J_{\nu-k+1}(ut)\}$$

$$= \int_0^\infty u^k J_\nu(\rho u)\, du \left[t^{\alpha+1} f(t) J_{\nu-k+1}(ut) - \int \frac{d}{dt}\{f(t) t^{\alpha-\nu+k}\} t^{\nu-k+1} J_{\nu-k+1}(ut)\, dt \right]_0^1$$

and, on now inverting the order of integration in the second term, this expression vanishes when $\rho > 1$, $\mathrm{re}(k) < 1$, $\mathrm{re}(\nu) > -1$, since

$$\int_0^\infty u^k J_{\nu-k+1}(ut) J_\nu(\rho u)\, du$$

vanishes whenever $\rho > t$, Watson (3a); since $\rho > 1$ and $0 \leqslant t \leqslant 1$, this condition is satisfied.

4. Condition that equation (1) is satisfied

We give first two lemmas without proof:

LEMMA 1. *If* $x = \rho \sin\phi$, $0 < \mathrm{re}(k) < 1$ *and*

$$I(x) = \int_0^1 t^{\nu+1}(1-t^2)^{-k}\{(\nu-2k+2)h(xt) + xth'(xt)\}\, dt \tag{5}$$

then $\quad \int_0^{\frac{1}{2}\pi} x^{\nu-2k+1} I(x)\cos^{2k-1}\phi\, d\phi = \frac{1}{2}\Gamma(k)\Gamma(1-k)\rho^{\nu-2k+1}h(\rho)$,

provided that $h(\rho)$ *is suitably restricted.*

The method of proof of Lemma 1 is quite straightforward. It is to write $xt = y$ in the second integral and then to invert the order of integration.

LEMMA 2. *If* $x = \rho\sin\phi$ *and* $F(x)$ *is continuous at or near* $x = +0$, *and does not change sign infinitely often near* $x = +0$, *and if*

$$\int_0^{\frac{1}{2}\pi} F(x)\cos^{2k-1}\phi\, d\phi = 0$$

for all ρ *such that* $0 < \rho < 1$, *then* $F(x) \equiv 0$.

The proof of this lemma follows the usual method for such problems.

If we could take $a^{-1}u^{-2k}$, where a is a constant, as an approximation to $G(u)$, the pair of equations (1) and (2) would be those of Busbridge (**4**) and the solution could be written down. Accordingly we write

$$G(u) = a^{-1}u^{-2k} + G^*(u)$$

and attempt to find an a and a k so that $G^*(u)$ is small. Writing $G(u)$ in the form just given and inserting the value of $g(u)$ given by (3) into equation (1) we obtain, on multiplying by a, the equation

$$I_1 + I_2 = ah(\rho),$$

where

$$I_1 = A\,2^{k-1}\Gamma(k)\int\limits_0^\infty aG^*(u)u^{1+k}J_\nu(\rho u)\,du\int\limits_0^1 f(t)t^{\alpha+1}J_{\nu-k}(ut)\,dt,$$

$$I_2 = A\,2^{k-1}\Gamma(k)\int\limits_0^\infty u^{1-k}J_\nu(\rho u)\,du\int\limits_0^1 f(t)t^{\alpha+1}J_{\nu-k}(ut)\,dt.$$

We invert the order of integration in I_2, and by Watson (**3** a) the u integral vanishes if $t > \rho$, and is equal to

$$\frac{t^{\nu-k}\rho^{-\nu+2k-2}}{2^{k-1}\Gamma(k)}\,{}_2F_1(\nu-k+1, 1-k; \nu-k+1; t^2/\rho^2),$$

if $t < \rho$. Noting that the hypergeometric function is equal to $(1-t^2/\rho^2)^{k-1}$, and that the t integration now runs from 0 to ρ, we obtain on writing $t = x$ and then $x = \rho\sin\phi$

$$I_2 = A\int\limits_0^{\frac12\pi}\rho^{-\nu+2k-1}f(x)x^{\alpha+\nu-k+1}\cos^{2k-1}\phi\,d\phi.$$

We now consider I_1. On making use of Sonine's formula for J_ν in terms of $J_{\nu-k}$ (Watson (**3** b)) we obtain if $\mathrm{re}(\nu-k) > -1$, $\mathrm{re}(k) > 0$,

$$I_1 = A\int\limits_0^\infty du\int\limits_0^{\frac12\pi}d\phi\int\limits_0^1 aG^*(u)u^{2k+1}\rho^k f(t)t^{\alpha+1}\times$$

$$\times J_{\nu-k}(ut)J_{\nu-k}(u\rho\sin\phi)\sin^{\nu-k+1}\phi\cos^{2k-1}\phi\,dt.$$

On writing $\sin\phi = x/\rho$ in this integral we obtain, using the fact that $I_1 + I_2 = ah(\rho)$, multiplying through by $\rho^{\nu-2k+1}$ and inverting the order of integration,

$$A\int\limits_0^{\frac12\pi}\Bigg[f(x)x^{\alpha+\nu-k+1} +$$

$$+ x^{\nu-k+1}\int\limits_0^1 f(t)t^{\alpha+1}\,dt\int\limits_0^\infty aG^*(u)u^{2k+1}J_{\nu-k}(ut)J_{\nu-k}(ux)\,du\Bigg]\cos^{2k-1}\phi\,d\phi$$

$$= a\rho^{\nu-2k+1}h(\rho),$$

where $x = \rho\sin\phi$.

We write

$$K(x,t) = \int_0^\infty aG^*(u)u^{2k+1}J_{\nu-k}(ut)J_{\nu-k}(ux)\,du$$

$$= \int_0^\infty \{au^{2k}G(u)-1\}uJ_{\nu-k}(ut)J_{\nu-k}(ux)\,du. \tag{6}$$

Hence by Lemma 1 and equation (4) we have

$$A\int_0^{\frac12\pi} \left\{ f(x)x^{\alpha+\nu-k+1}+x^{\nu-k+1}\int_0^1 K(x,t)f(t)t^{\alpha+1}\,dt -ax^{\nu-2k+1}I(x)\right\}\cos^{2k-1}\phi\,d\phi$$
$$= 0,$$

and so we see, using Lemma 2 and multiplying by $x^{-\alpha-\nu+k-1}$, that $f(x)$ satisfies the integral equation

$$f(x)+x^{-\alpha}\int_0^1 K(x,t)t^{\alpha+1}f(t)\,dt = ax^{-\alpha-k}I(x),$$

where $K(x,t)$ and $I(x)$ are given by equations (6) and (5) respectively.

The case $h(\rho) = \rho^\nu$ is the most important in applications. It is convenient to transfer the constants in the solution in this case and we shall write it out in full for later reference. It is

$$g(u) = \frac{2^k\Gamma(\nu+1)}{\Gamma(\nu-k+1)}u^{1+k}\int_0^1 f(t)t^{\alpha+1}J_{\nu-k}(ut)\,dt,$$

where $f(x)$ satisfies the integral equation

$$f(x)+x^{-\alpha}\int_0^1 t^{\alpha+1}f(t)\,dt\int_0^\infty \{au^{2k}G(u)-1\}uJ_{\nu-k}(ut)J_{\nu-k}(ux)\,du = ax^{\nu-\alpha-k}.$$

In the solutions so far given a, α, and k are at our disposal; except that k must satisfy the conditions $\mathrm{re}(\nu-k) > -1$ and $0 < \mathrm{re}(k) < 1$.

5. Choice of a and k

The idea is to find a and k so that $G^*(u)$ is small, or in other words so that $G(u)$ is closely approximated by $a^{-1}u^{-2k}$. It is in any case necessary that the integral for $K(x,t)$ in equation (6) should converge and this seems to determine the choice of a and k in any given problem. It is obvious that this requires the conditions on $G(u)$ to be fairly severe. If this integral converges absolutely and $G^*(u)$ is continuous, the inversions of order of integration given above are legitimate, but there are almost certainly conditions less severe than the absolute convergence of the integral. It would be a very difficult task to specify them completely.

6. The case $k < 0$

It seems fairly certain that the solution given is valid when $\operatorname{re}(k) < 1$ and not merely when $0 < \operatorname{re}(k) < 1$, provided that the functions are suitably restricted, but I have been unable to prove this. In the applications given here the relevant value of k is $\frac{1}{2}$ in every case. I have, however, tried out the diffraction problem solved by Tranter (2), in which k must be given the value $-\frac{1}{2}$, and have obtained a solution exactly the same as his.

If the solution is valid for $k < 0$ then equation (5) for $I(x)$ may be rewritten as

$$I(x) = -2k \int\limits_0^1 h(xt)t^{\nu+1}(1-t^2)^{-k-1}\, dt,$$

as is easily seen.

We have thus reduced the problem to the solution of a Fredholm's integral equation, which will not suffer from the disadvantages of Tranter's linear equations in an infinite number of unknowns. In most cases the u integration must be performed numerically, but nevertheless a solution is practically possible which was not always the case in Tranter's solution.

7. Applications

1. *Two parallel equal coaxial disks*

The disks are taken to have unit radius and to be situated in the planes $z = 0$ and $z = h$, where ρ, ϕ, and z are cylindrical polar coordinates. Owing to axial symmetry ϕ does not appear in the equations.

We shall consider two problems at once. The first is two equally (or equally and oppositely) charged circular disks and the second is two circular disks rotating slowly in a viscous fluid with equal (or equal and opposite) angular velocities, the magnitude of the potential or angular velocity being V_0. In the latter problem the Stokes approximation is used.

The solution is

$$V = V_0 \int\limits_0^\infty \frac{e^{-|z|u} \pm e^{-|h-z|u}}{u}\, g(u) J_\nu(\rho u)\, du,$$

the sign being positive or negative according as the disks are of like or unlike potential or angular velocity V_0.

In the electrical case $\nu = 0$, in the hydrodynamical case $\nu = 1$. See Jeffery (5). This form satisfies the conditions at infinity and also the relevant differential equations in the two cases. It satisfies the conditions on the disks if

$$\int\limits_0^\infty g(u)(1 \pm e^{-hu})u^{-1}J_\nu(\rho u)\, du = \rho^\nu \quad (0 < \rho < 1).$$

The *form* for V is discontinuous on $z = 0$ and $z = h$. However, for $\rho > 1$ it is necessary that for $z = 0$ and $z = h$ V be continuous and also $\partial V/\partial z$. The first condition is satisfied by the form for V; the second is satisfied if equation (2) holds. Alternatively, to see that equation (2) holds we may say in the electrical case that, as the *form* for V is discontinuous on the part of the plane $z = 0$, $\rho > 1$, this discontinuity must not be represented by any distribution of charge on this plane and hence the expression (7) below for the surface density must vanish when $\rho > 1$.

We take $k = \frac{1}{2}$, $\alpha = -\frac{1}{2}$, $a = 1$ and the solution is

$$g(u) = \frac{2^{\frac{1}{2}}\Gamma(\nu+1)}{\Gamma(\nu+\frac{1}{2})} u^{\frac{1}{2}} \int_0^\infty f(t) t^{\frac{1}{2}} J_{\nu-\frac{1}{2}}(ut)\, dt,$$

where
$$f(x) \pm \frac{2}{\pi} \int_0^1 f(t)\, dt \int_0^\infty e^{-hu} \frac{\cos ut \cos ux}{\sin ut \sin ux}\, du = x^\nu,$$

and the cosines are taken for $\nu = 0$ and the sines for $\nu = 1$. It can be shown that $f(x)$ is an even function for $\nu = 0$ and an odd function for $\nu = 1$ and the integral equation may be written

$$f(x) \pm (1/\pi) \int_{-1}^1 f(t)\, dt \int_0^\infty e^{-hu} \cos(x-t)u\, du = x^\nu$$

in either case, or

$$f(x) \pm \frac{1}{\pi} \int_{-1}^1 f(t) \frac{h}{h^2+(x-t)^2}\, dt = x^\nu.$$

When $\nu = 0$ this is Love's integral equation solution of the problem, (6), although his form for V is different. The result for $\nu = 1$ seems to be new. The general solution of equations (1) and (2) of this paper was in fact first discovered by a consideration of Love's solution of the two-disk problem.

2. Disk between two parallel planes

We distinguish two cases as before, namely the charged disk and the disk rotating slowly in fluid. We take the disk of radius unity in the plane $z = 0$, and the two parallel planes to be $z = \lambda$ and $z = -\mu$. The planes are supposed to be at zero potential or at rest respectively.

Consideration of the image system leads to the value

$$V = V_0 \int_0^\infty \frac{2\sinh\mu u \sinh(\lambda-z)u}{u\sinh(\lambda+\mu)u} g(u) J_\nu(\rho u)\, du$$

for $0 \leqslant z \leqslant \lambda$; when $-\mu \leqslant z \leqslant 0$, the term $2\sinh\mu u \sinh(\lambda-z)u$ must be

replaced by $2\sinh\lambda u\sinh(\mu+z)u$. This form makes V vanish when $z=\lambda$ or $z=-\mu$.

The equations to be solved are equations (1) and (2) with $h(\rho)=\rho^\nu$, and

$$G(u)=\frac{2\sinh\lambda u\sinh\mu u}{u\sinh(\lambda+\mu)\mu}.$$

Equation (2) arises from the same considerations as given in Application 1. The solution is

$$g(u)=\frac{2^{\frac{1}{2}}\Gamma(\nu+1)}{\Gamma(\nu+\frac{1}{2})}u^{\frac{1}{2}}\int_0^\infty f(t)t^{\frac{1}{2}}J_{\nu-\frac{1}{2}}(ut)\,dt,$$

where

$$f(x)+\int_0^1(xt)^{\frac{1}{2}}f(t)\,dt\int_0^\infty\{uG(u)-1\}uJ_{\nu-\frac{1}{2}}(ut)J_{\nu-\frac{1}{2}}(ux)\,du=x^\nu.$$

As before we can show that $f(x)$ is even for $\nu=0$ and odd for $\nu=1$, and the integral equation may be written

$$f(x)-(1/\pi)\int_{-1}^1f(t)\,dt\int_0^\infty\{1-uG(u)\}\cos(x-t)u\,du=x^\nu.$$

In the case $\lambda=\mu=h$ the problem reduces to that of Tranter (1) and the integral equation to be satisfied is

$$f(x)-(1/\pi)\int_{-1}^1f(t)\,dt\int_0^\infty(1-\tanh hu)\cos(x-t)u\,du=x^\nu.$$

It can be shown fairly simply that in the case $\nu=0$, h large, this equation gives the same results as Tranter's method, but it can be used even if h is not large. The infinite integral must be evaluated numerically, but this is not difficult if we write $uh=y$ and use Filon's method, (8), (7 b), even if h is small. If $(x-t)/h$ is large an asymptotic form for the integral may be used. See Tranter (7 a).

3. *Charge or turning moment on a disk*

The surface density or frictional drag per unit area on the positive side of the disk $z=0$ is $-K(\partial V_+/\partial z)_{z=+0}$, and on the negative side it is $-K(\partial V_-/\partial(-z))_{z=-0}$, where V_+ and V_- are the values of V on the positive and negative sides respectively. K has the value $1/4\pi$ or μ (the coefficient of viscosity), as the case may be.

Adding together the values on each side, we obtain in all the cases considered here the value

$$2KV_0\int_0^\infty g(u)J_\nu(\rho u)\,du \tag{7}$$

for the surface density or frictional drag.

Hence the total charge or turning couple is

$$2KV_0 \int\limits_0^1 2\pi\rho^{\nu+1}\, d\rho \int\limits_0^\infty g(u)J_\nu(\rho u)\, du,$$

and on inserting the value of $g(u)$ and integrating with respect to ρ and u in that order we obtain in a straightforward manner

$$\frac{8\pi KV_0\Gamma^2(\nu+1)}{\Gamma^2(\nu+\tfrac{1}{2})} \int\limits_0^1 t^\nu f(t)\, dt,$$

for the charge or turning couple.

Writing $K = 1/4\pi$, $\nu = 0$ we have for the total charge on a disk

$$(2V_0/\pi) \int\limits_0^1 f(t)\, dt,$$

which is Love's result; writing $K = \mu$, $\nu = 1$ we have for the turning couple on a disk

$$32\mu V_0 \int\limits_0^1 tf(t)\, dt.$$

In the case of a single charged or rotating disk, in which $f(x) = x^\nu$, the values are $2V_0/\pi$ and $32\mu V_0/3$ respectively, as is well known.

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ROUND-OFF ERRORS IN IMPLICIT FINITE DIFFERENCE METHODS

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SUMMARY

Symmetrical and asymmetrical implicit finite difference replacements involving a variable parameter a and a variable mesh ratio s are considered for the heat conduction and wave equations, and expressions obtained for the round-off errors.

It is found that the stable backward difference replacements, four-point for the heat conduction equation and five-point for the wave equation, give rise to minimum round-off errors.

1. Introduction

IMPLICIT finite difference approximations have been used by several authors, Crank and Nicolson (**1**) and Hartree and Womersley (**2**) *inter alia*. The stability advantages of such approximations to parabolic and hyperbolic differential equations were first pointed out by von Neumann. Implicit approximations, however, almost always involve solutions which are relaxational in one of the independent variables. As a result, such solutions require considerable labour and are liable to involve substantial round-off errors.

In the present paper, general implicit finite difference replacements involving a variable parameter are considered for the heat conduction and wave equations, with two independent variables and normal boundary conditions in each case. Provided closed expressions can be obtained for the round-off errors, the values of the parameter and the mesh ratio giving rise to minimum round-off errors will be determined for any specific implicit finite difference replacement.

The present author (**3**, **4**) has already obtained expressions for the round-off errors arising in relaxational solutions of Poisson's equation and the heat conduction equation, von Neumann's stable six-point formula being used in the latter case.

2. The heat conduction equation

The simplest form of the parabolic heat conduction equation is

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial \phi}{\partial t}, \tag{1}$$

where $\phi(x, t)$ is the temperature and x and t are the distance and time coordinates respectively. The boundary conditions consist of a knowledge

of ϕ along $x = 0$, L and $t = 0$. The solution is required in the region $0 \leqslant x \leqslant L$, $t \geqslant 0$.

Suppose this region is covered by a rectangular net, the mesh lengths being Δx and Δt in the x- and t-directions respectively. If j and k are the row and column numbers respectively, a general implicit finite difference replacement of (1) is

$$a[\phi_{j,k-1}-2\phi_{j,k}+\phi_{j,k+1}]+(1-a)[\phi_{j-1,k-1}-2\phi_{j-1,k}+\phi_{j-1,k+1}]$$
$$= \frac{(\Delta x)^2}{(\Delta t)}[\phi_{j,k}-\phi_{j-1,k}] \quad (j = 1, 2,...; k = 1, 2,..., N), \quad (2)$$

where $a > 0$. This difference equation is first order and step by step in the time coordinate and second order and relaxational in the distance coordinate. Using this equation along with a knowledge of ϕ at all nodes on $x = 0$, L and $t = 0$, a relaxational solution can be obtained for each row in turn. For the case of $(N+2)$ columns, denoting the residuals by $r_{j,k}$ $(j = 1, 2,...; k = 1, 2,..., N)$, the errors $\epsilon_{j,k}$ in $\phi_{j,k}$ satisfy the error equations

$$\left[\epsilon_{j,k-1}-\left(2+\frac{s}{a}\right)\epsilon_{j,k}+\epsilon_{j,k+1}\right]+$$
$$+\frac{1-a}{a}\left[\epsilon_{j-1,k-1}-\left(2-\frac{s}{1-a}\right)\epsilon_{j-1,k}+\epsilon_{j-1,k+1}\right] = r_{j,k}, \quad (3)$$

where the mesh ratio s is given by $s = (\Delta x)^2/(\Delta t)$. Suppose that no rounding off is required at nodes on the boundaries, giving $\epsilon_{j,0} = \epsilon_{j,N+1} = \epsilon_{0,k} = 0$ for all j and k.

If $\mathbf{R}_1 = \mathbf{R}_1$, $\mathbf{R}_j = 0$ $(j = 2, 3,...)$, the growth of error is given by

$$\mathbf{E}_j = (-\mathbf{A}^{-1}\mathbf{B})^{j-1}\mathbf{A}^{-1}\mathbf{R}_1 \quad (j = 1, 2,...), \quad (4)$$

where

$$\mathbf{E}_j = \begin{bmatrix} \epsilon_{j,1} \\ \cdot \\ \cdot \\ \cdot \\ \epsilon_{j,N} \end{bmatrix}, \quad \mathbf{R}_j = \begin{bmatrix} r_{j,1} \\ \cdot \\ \cdot \\ \cdot \\ r_{j,N} \end{bmatrix} \quad (j = 1, 2,...),$$

$\mathbf{A} = \mathbf{P}\left(2+\frac{s}{a}\right)$, and $\mathbf{B} = \frac{1-a}{a}\mathbf{P}\left(2-\frac{s}{1-a}\right)$, where $\mathbf{P}(x)$ denotes the square matrix

$$\begin{bmatrix} -x & 1 & & & \\ 1 & -x & 1 & & \\ & & \cdot & & \\ & & & \cdot & \\ & & 1 & -x & 1 \\ & & & 1 & -x \end{bmatrix}$$

of order N. Now the latent roots of $\mathbf{A}^{-1}\mathbf{B}$ are

$$-1+\frac{4\cos^2\dfrac{\alpha\pi}{2(N+1)}}{s+4a\cos^2\dfrac{\alpha\pi}{2(N+1)}}\quad(\alpha=1,...,N),$$

all of which have modulus less than or equal to unity for $s \geqslant 2(1-2a)$. Thus (2) is shown to be stable for all s if $a \geqslant \frac{1}{2}$ and for $s \geqslant 2(1-2a)$ if $0 < a < \frac{1}{2}$, a result which can also be obtained using von Neumann's criterion for stability (see ref. 5).

The magnitude of the round-off errors will of course depend on the residual distribution. A good indication of the variation of the errors with s and a can be obtained by considering the variation of the maximum errors with these parameters. Consider first of all the residual distribution $\mathbf{R}_j = \mathbf{R}_1$ $(j = 2, 3,...)$. Using (4) it is easily shown that as the row number increases the error vector approaches the value

$$\mathbf{e}_1 = (\mathbf{A}+\mathbf{B})^{-1}\mathbf{R}_1 = a[\mathbf{P}(2)]^{-1}\mathbf{R}_1. \tag{5}$$

Next consider the residual distribution $\mathbf{R}_j = (-1)^{j+1}\mathbf{R}_1$ $(j = 2, 3,...)$. This gives rise to an error vector whose value approaches

$$\mathbf{e}_2 = (\mathbf{A}-\mathbf{B})^{-1}\mathbf{R}_1$$

$$= \frac{a}{2a-1}\left[\mathbf{P}\left\{2\left(1+\frac{s}{2a-1}\right)\right\}\right]^{-1}\mathbf{R}_1 \quad (a \neq \tfrac{1}{2})$$

$$= \frac{1}{4s}\mathbf{R}_1 \qquad\qquad (a = \tfrac{1}{2}) \tag{6}$$

where no importance is attached to the sign of $\mathbf{e}_2$. The values of $[\mathbf{P}(x)]^{-1}$ are given by Rutherford (6). The maximum possible error vector is given by (5) or (6) with $r_{1,k} = r$ $(1 \leqslant k \leqslant N)$. This is shown by the present author (4) for the case $a = \frac{1}{2}$.

For $a \geqslant 1$, the maximum error vector is given by $\mathbf{e}_1$ which is independent of s. From (5) it is easily seen that for this range of the parameter, the smallest maximum error is given by $a = 1$. The value of this error is $[\mathbf{P}(2)]^{-1}\mathbf{R}_1$, which is independent of the mesh ratio s, but is a function of the number of internal columns N. For $\frac{1}{2} \leqslant a < 1$, the maximum error vector is given by either $\mathbf{e}_1$ or $\mathbf{e}_2$ depending on the values of a, s, and N. For $0 < a < \frac{1}{2}$, the maximum error is given by $\mathbf{e}_2$ for all s and N.

Without carrying out detailed calculations in the range $0 < a < 1$, it can be said that if stability is required for all s, the value of a giving rise to minimum round-off errors always lies in the range $\frac{1}{2} \leqslant a \leqslant 1$. When $a = \frac{1}{2}$, (2) becomes the six point formula suggested by von Neumann, the round-off errors of which have been studied in detail by the present author

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(4). When $a = 1$, (2) becomes the convenient backward four-point formula with the simple result that the maximum round-off error is $[\mathbf{P}(2)]^{-1}\mathbf{R}_1$ for all values of the mesh ratio s.

3. The wave equation

The wave equation is considered in the form

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial t^2}, \tag{7}$$

where $\phi(x,t)$ is the displacement. The boundary conditions consist of a knowledge of ϕ along $x = 0, L$, and of ϕ and $\partial\phi/\partial t$ along $t = 0$. Again the solution is required in the region $0 \leqslant x \leqslant L$, $t \geqslant 0$, which is covered by a rectangular net of mesh lengths Δx and Δt. Two general implicit finite difference replacements of (7) are considered. Both replacements are second order and step by step in the time coordinate and relaxational in the distance coordinate. Using either difference equation along with the values of ϕ at all nodes on $x = 0, L$ and $t = 0, \Delta t$ a relaxational solution can be obtained for each row in turn.

(a) The asymmetrical implicit finite difference replacement

A general asymmetrical implicit finite difference replacement of (7) is

$$a[\phi_{j,k-1} - 2\phi_{j,k} + \phi_{j,k+1}] + (1-a)[\phi_{j-2,k-1} - 2\phi_{j-2,k} + \phi_{j-2,k+1}]$$
$$= \left(\frac{\Delta x}{\Delta t}\right)^2 [\phi_{j,k} - 2\phi_{j-1,k} + \phi_{j-2,k}] \quad (j = 2, 3, \ldots;\ k = 1, 2, \ldots, N), \tag{8}$$

where $a > 0$. The errors $\epsilon_{j,k}$ in $\phi_{j,k}$ satisfy the equations

$$\left[\epsilon_{j,k-1} - \left(2 + \frac{s^2}{a}\right)\epsilon_{j,k} + \epsilon_{j,k+1}\right] + 2\frac{s^2}{a}\epsilon_{j-1,k} +$$
$$+ \frac{1-a}{a}\left[\epsilon_{j-2,k-1} - \left(2 + \frac{s^2}{1-a}\right)\epsilon_{j-2,k} + \epsilon_{j-2,k+1}\right] = r_{j,k}$$
$$(j = 2, 3, \ldots;\ k = 1, 2, \ldots, N), \tag{9}$$

where $s = \Delta x/\Delta t$ is the mesh ratio. Again for convenience suppose that no rounding off is required at nodes on $x = 0, L$ and $t = 0, \Delta t$. Hence $\epsilon_{j,0} = \epsilon_{j,N+1} = \epsilon_{0,k} = \epsilon_{1,k} = 0$ for all j and k. If rounding is required at such nodes, and it will probably be necessary at nodes on $t = \Delta t$, the errors neglected will have small effect on the maximum round-off error.

It is easily shown, using von Neumann's method of examining stability, that (8) is stable for all s if $a \geqslant \frac{1}{2}$, and unstable for all s if $0 < a < \frac{1}{2}$. Again from (9) the error vector in the row $j = 2$ is given by

$$\mathbf{E}_2 = \mathbf{A}^{-1}\mathbf{R}_2. \tag{10}$$

where $A = P(2+s^2/a)$. Now the latent roots of A^{-1}, given by

$$\left(4\cos^2\frac{\alpha}{N+1}\frac{\pi}{2}+\frac{s^2}{a}\right)^{-1} \quad (\alpha = 1, 2,..., N),$$

are small when s^2/a is large. Thus the value $a = \frac{1}{2}$ ensures stability for all s and gives rise to minimum round-off errors in the row $j = 2$.

Making this simplification $a = \frac{1}{2}$ in (9) and putting $R_2 = R_2$, $R_j = 0$ ($j = 3, 4,...$) the growth of error as far as $j = 12$ is given by Table 1 where $A = P[2(1+s^2)]$ and $B = 4s^2I$, I being the unit matrix of order N. The main feature of the table is the occurrence of the binomial coefficients in the diagonal columns.

TABLE 1

	E_2	$(A^{-1}B)E_2$	$(A^{-1}B)^2E_2$	$(A^{-1}B)^3E_2$	$(A^{-1}B)^4E_2$	$(A^{-1}B)^5E_2$	$(A^{-1}B)^6E_2$	$(A^{-1}B)^7E_2$	$(A^{-1}B)^8E_2$	$(A^{-1}B)^9E_2$	$(A^{-1}B)^{10}E_2$
E_2	1										
E_3		-1									
E_4	-1		1								
E_5		2		-1							
E_6	1		-3		1						
E_7		-3		4		-1					
E_8	-1		6		-5		1				
E_9		4		-10		6		-1			
E_{10}	1		-10		15		-7		1		
E_{11}		-5		20		-21		8		-1	
E_{12}	-1		15		-35		28		-9		1

Now the latent roots of $A^{-1}B$ are

$$2\left(1+\frac{2}{s^2}\cos^2\frac{\alpha}{N+1}\frac{\pi}{2}\right)^{-1} \quad (\alpha = 1, 2,..., N),$$

all of which tend to zero as the mesh ratio s tends to zero. Thus for $s = 0$, it is seen from Table 1 that $E_2 = -E_4 = E_6 = -E_8 = ... = A^{-1}R_2$ and $E_3 = E_5 = E_7 = ... = 0$. This value of the mesh ratio together with the residual distribution $R_j = (-1)^{j/2+1}R_2$ ($j = 2, 4,...$) gives rise to error vectors

$$e_j = e_{j+1} = \frac{j}{2}(A^{-1}R_2) \quad (j = 2, 4,...), \tag{11}$$

where $A^{-1} = [P(2)]^{-1}$. The value $s = 0$ is of course quite artificial, but the error vector obtained provides a useful guide to the maximum error possible for small s. This is illustrated in Fig. 1 where the growth of error ratio and the maximum error possible at the middle node of each row as far as $j = 20$ are shown for $s = 1, 0.1$, and 0 when $N = 3$. The growth of error ratio at any row is considered to be the ratio of the error at the middle

node in that row to the magnitude of the error at the middle node in the row $j = 2$.
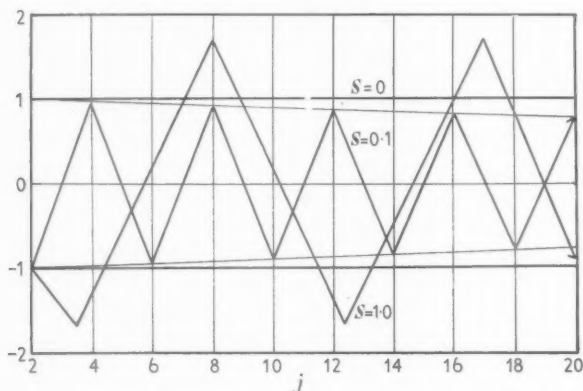
FIG. 1 a
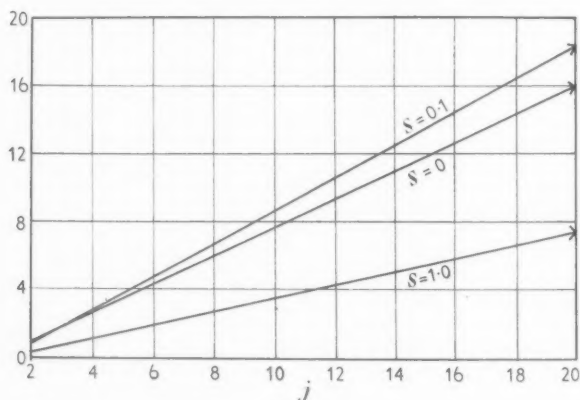
FIG. 1 b

Another useful value of the parameter is $a = 1$. The modified equation (9) together with the residual distribution $\mathbf{R}_2 = \mathbf{R}_2$, $\mathbf{R}_j = 0$ ($j = 3, 4,...$) leads to an error growth which is given by Table 2 as far as $j = 12$, where $\mathbf{A} = \mathbf{P}(2+s^2)$.

Now the latent roots of $s^2\mathbf{A}^{--}$ are

$$\left(1+\frac{4}{s^2}\cos^2\frac{\alpha}{N+1}\frac{\pi}{2}\right)^{-1} \quad (\alpha = 1, 2,..., N),$$

TABLE 2

	E_2	$(^sA^{-1})E_2$	$(^sA^{-1})^2E_2$	$(^sA^{-1})^3E_2$	$(^sA^{-1})^4E_2$	$(^sA^{-1})^5E_2$	$(^sA^{-1})^6E_2$	$(^sA^{-1})^7E_2$	$(^sA^{-1})^8E_2$	$(^sA^{-1})^9E_2$	$(^sA^{-1})^{10}E_2$
E_2	1										
E_3		−2									
E_4		1	4								
E_5			−4	−8							
E_6			1	12	16						
E_7				−6	−32	−32					
E_8				1	24	80	64				
E_9					−8	−80	−192	−128			
E_{10}					1	40	240	448	256		
E_{11}						−10	−224	−512	−1024	−512	
E_{12}						1	124	400	1792	2304	1024

all of which tend to zero as s tends to zero. Thus for $s = 0$ it is seen from Table 2 that $\mathbf{E}_3 = \mathbf{E}_4 = \mathbf{E}_5 = ... = 0$, and so irrespective of the values of the residuals at nodes for which $j > 2$, the error vectors are given by

$$\mathbf{e}_j = \mathbf{A}^{-1}\mathbf{R}_2 \quad (j = 2, 3, 4,...). \tag{12}$$

Once again the value $s = 0$ is completely artificial, but the error vector given by (12) provides an indication of the size of the maximum error possible for small s. The growth of error ratio and the maximum error possible are shown in Fig. 2 for $s = 1$, 0·1, and 0 when $N = 3$.

Comparing the errors given in Figs. 1 and 2, it is seen that although $a = \frac{1}{2}$ gives rise to minimum errors in the row $j = 2$, $a = 1$ gives a much smaller error growth. In fact, when $s \leqslant 1$, no matter how many rows of calculation are considered, there is no chance of large round-off errors using the asymmetrical replacement (8) with $a = 1$, provided N is not excessively large.

(b) *The symmetrical implicit finite difference replacement*

A general symmetrical implicit finite difference replacement of (7) introduced by von Neumann (5) is

$$a[\phi_{j,k-1} - 2\phi_{j,k} + \phi_{j,k+1}] + (1-2a)[\phi_{j-1,k-1} - 2\phi_{j-1,k} + \phi_{j-1,k+1}] +$$
$$+ a[\phi_{j-2,k-1} - 2\phi_{j-2,k} + \phi_{j-2,k+1}] = \left(\frac{\Delta x}{\Delta t}\right)^2 [\phi_{j,k} - 2\phi_{j-1,k} + \phi_{j-2,k}]$$
$$(j = 2, 3,...; \ k = 1, 2,..., N) \tag{13}$$

where $a > 0$. The finite difference replacements (8) and (13) reduce to the same expression when $a = \frac{1}{2}$, so this value of a will be excluded from the
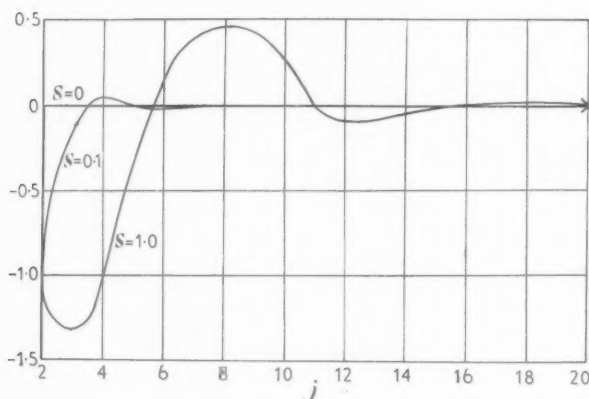
Fig. 2 a
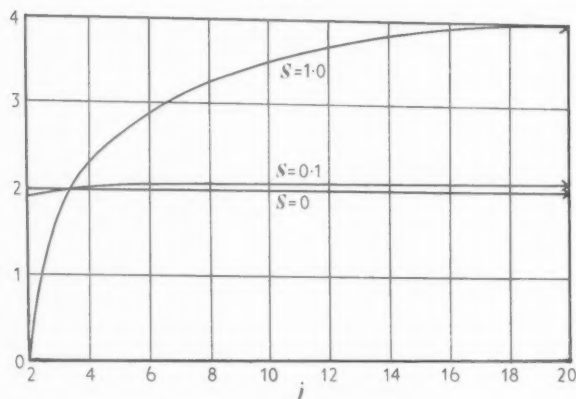
Fig. 2 b

present section. The errors $\epsilon_{j,k}$ in $\phi_{j,k}$ satisfy the equations

$$\left[\epsilon_{j,k-1}-\left(2+\frac{s^2}{a}\right)\epsilon_{j,k}+\epsilon_{j,k+1}\right]+$$
$$+\left(\frac{1-2a}{a}\right)\left[\epsilon_{j-1,k-1}-2\left(1-\frac{s^2}{1-2a}\right)\epsilon_{j-1,k}+\epsilon_{j-1,k+1}\right]+$$
$$+\left[\epsilon_{j-2,k-1}-\left(2+\frac{s^2}{a}\right)\epsilon_{j-2,k}+\epsilon_{j-2,k+1}\right]=r_{j,k}$$
$$(j=2, 3,...;\ k=1, 2,..., N). \quad (14)$$

As with the asymmetrical replacement, suppose that no rounding off is required at nodes on $x = 0$, L and $t = 0$, Δt. This time (13) is stable for all $s \geqslant 0$ if $a > \frac{1}{4}$ and stable for $s > (1-4a)^{\frac{1}{2}}$ if $a \leqslant \frac{1}{4}$.

The growth of error in the symmetrical case is given by Table 1 with

$$\mathbf{A} = \mathbf{P}\left(2 + \frac{s^2}{a}\right) \quad \text{and} \quad \mathbf{B} = \left(\frac{1-2a}{a}\right)\mathbf{P}\left[2\left\{1 - \frac{s^2}{1-2a}\right\}\right].$$

The latent roots of $\mathbf{A}^{-1}\mathbf{B}$ are

$$\frac{\left(\dfrac{1-2a}{a}\right)\left[2\cos^2\dfrac{\alpha\pi}{2(N+1)} - \dfrac{s^2}{1-2a}\right]}{\left[2\cos^2\dfrac{\alpha\pi}{2(N+1)} + \dfrac{s^2}{2a}\right]} \quad (\alpha = 1, 2, ..., N),$$

all of which tend to $\dfrac{1-2a}{a}$ as s tends to zero, and to -2 as s tends to infinity.

In addition, it can easily be shown that the magnitude of a latent root exceeds unity, for

$$s^2 \geqslant 0 \quad (a \geqslant 1), \qquad s^2 > 4(1-a)\cos^2\frac{\alpha\pi}{2(N+1)} \quad (\tfrac{1}{2} < a < 1),$$

and

$$s^2 > 4(1-a)\cos^2\frac{\alpha\pi}{2(N+1)} \quad \text{or} \quad s^2 < \tfrac{4}{3}(1-3a)\cos^2\frac{\alpha\pi}{2(N+1)} \quad (a < \tfrac{1}{2}).$$

The errors in the row $j = 2$ are again given by (10). For minimum round-off error a is required to be as small as possible and the smallest value for which there is stability for all s is $a = \tfrac{1}{4}$. With $a = \tfrac{1}{4}$ in (14), the growth of error is given as far as $j = 12$ by Table 1 with $\mathbf{A} = \mathbf{P}\{2(1+2s^2)\}$ and $\mathbf{B} = 2\mathbf{P}\{2(1-2s^2)\}$. Here the latent roots of $\mathbf{A}^{-1}\mathbf{B}$ tend to 2 as s tends to zero, and their magnitudes are all less than unity when s lies within the range $\dfrac{1}{6}\left(1 + \cos\dfrac{\pi}{N+1}\right) < s^2 < \dfrac{3}{2}\left(1 + \cos\dfrac{N\pi}{N+1}\right)$. It seems likely that the minimum error growth will occur for a value of s within the above range. As an illustration, consider the simple case $N = 3$ where the range reduces to $0.55 < s < 0.65$. The growth of error ratio and the maximum error possible at the middle node of each row as far as $j = 20$ are shown for $s = 1, 0.6$, and 0 in Fig. 3. The artificial case $s = 0$ being unstable provides a limiting value for the error growth which is far in excess of the growth when s is small but not zero. When $s = 0$, it is seen from Table 1 that $E_j = (-1)^j(j-1)E_2 \ (j \geqslant 2)$ and so using the residual distribution

$$\mathbf{R}_j = (-1)^j\mathbf{R}_2 \quad (j \geqslant 2),$$

the maximum error vectors become

$$\mathbf{e}_j = \frac{j(j-1)}{2}(\mathbf{A}^{-1}\mathbf{R}_2) \quad (j = 2, 3, ...). \tag{15}$$

Consider lastly the value of the parameter $a = 1$. The latent roots of $\mathbf{A}^{-1}\mathbf{B}$ tend to -1 as s tends to zero and to -2 as s tends to infinity, and
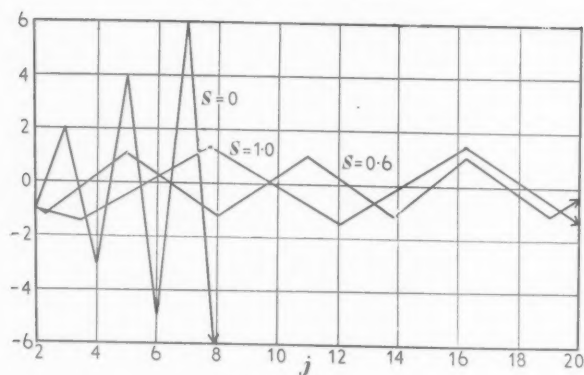
FIG. 3 a
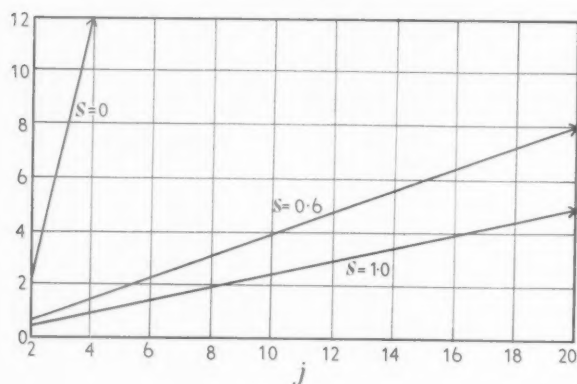
FIG. 3 b

so the minimum error growth will occur for small s. This time the case $s = 0$ is stable and so gives a useful guide to the errors for small s. With $s = 0$, $\mathbf{A} = -\mathbf{B} = \mathbf{P}(2)$, and so from Table 1,

$$\left.\begin{array}{l} \mathbf{E}_{3p-1} = (-1)^{p+1}\mathbf{E}_2 \\ \mathbf{E}_j \equiv \quad \mathbf{E}_{3p} \;= (-1)^{p+1}\mathbf{E}_2 \\ \mathbf{E}_{3p+1} = 0 \end{array}\right\} \quad (p = 1, 2,\ldots).$$

Using the residual distribution

$$\left.\begin{array}{l} \mathbf{R}_{3p-1} = (-1)^{p+1}\mathbf{R}_2 \\ \mathbf{R}_j \equiv \quad \mathbf{R}_{3p} \;= (-1)^{p+1}\mathbf{R}_2 \\ \mathbf{R}_{3p+1} = 0 \end{array}\right\} \quad (p = 1, 2,\ldots),$$

the maximum error vectors become

$$\mathbf{e}_j \equiv \left.\begin{aligned} \mathbf{e}_{3p-1} &= p(\mathbf{A}^{-1}\mathbf{R}_2) \\ \mathbf{e}_{3p} &= 2p(\mathbf{A}^{-1}\mathbf{R}_2) \\ \mathbf{e}_{3p+1} &= p(\mathbf{A}^{-1}\mathbf{R}_2) \end{aligned}\right\} \quad (p = 1, 2, \ldots). \tag{16}$$

4. Concluding remarks

It should be realized that the round-off errors described in this paper are in excess of the errors likely to be incurred in actual calculations. Nevertheless, the present study will give useful information concerning the values of the variable parameter and the mesh ratio, which give rise to minimum round-off errors for each type of implicit finite difference replacement considered.

Explicit difference replacements of the heat conduction and wave equations are also susceptible to the methods described in the present paper, and will in fact lead to much simpler expressions for the errors.

Finally, for a given number of nodes along $t = 0$, the smaller the value of the mesh ratio s, the fewer will be the number of rows necessary to solve the problem for a given period of time, and so from round-off error considerations alone there may be no lower limit to the optimum value of s. For small s, however, the solution of the difference equation will not be a good approximation to the solution of the differential equation, and so considerations other than minimizing the round-off errors will impose a lower limit on the mesh ratio.

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SPECIAL TYPES OF GROUP RELAXATION FOR SIMULTANEOUS LINEAR EQUATIONS

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SUMMARY

In this paper a systematic procedure for the liquidation of residuals in the solution of simultaneous linear equations by the relaxation method is suggested. This process consists in obtaining groups which keep one or more of the residuals unchanged. A suitable choice of a multiple of these groups may be used to liquidate any of the remaining residuals.

1. Introduction

IN the solution of simultaneous linear equations, the excellence of the relaxation method lies in its flexibility. There is no set procedure for liquidating the residuals; neither is it the spirit of the process. Practice and intuition are the two essential tools in the operation. It happens, however, that in a number of cases some special techniques (1) viz. block relaxation, group relaxation, method of multiplying factors, etc., can be used to speed up the liquidation of the residuals. In these operations, too, there are no definite procedures for choosing the groups or blocks employed to bring the residuals to proper proportions.

In this paper a definite procedure has been suggested for the gradual liquidation of the residuals. The rule does not in any way affect the flexibility of the method, as the operator is quite free to use or not to use, or partially to use the rule according to his needs and requirements. The use of this rule will at the same time lead to a sure liquidation of the residuals in a finite number of steps. The method is open to a number of variations some of which have been detailed in this paper.

2. Details of the process

Let us suppose, for the sake of clarity, that three linear simultaneous equations have to be solved for three unknowns (x, y, and z). It is possible with the technique of point relaxation to reduce one of the residuals to zero. Let us also suppose that the residual has been liquidated by giving x the proper differential increment. As a first step (after liquidation of one of the residuals), it is possible, by a choice of two variables x and z, interchanging their coefficients and giving a negative value to one of them, to form a group

to maintain the liquidated residual at its zero value, while giving differential increments to the other two residuals. As a second step, by a choice of the variables x and y we can do likewise, when it is possible to keep the same liquidated residual at its zero value, giving the other two residuals differen-

TABLE 1

Equations:

$$-12x+10y+8z+24 = 0 \quad \equiv F_1$$
$$10x-12y+6z+76 = 0 \quad \equiv F_2$$
$$8x+6y-12z+320 = 0 \quad \equiv F_3$$

OPERATION TABLE

	Δx	Δy	Δz	ΔF_1	ΔF_2	ΔF_3
a	1			−12	10	8
		1		10	−12	6
			1	8	6	−12

Interchanging the coefficients of x and z in F_1
and changing the sign for one of them

	2		3	0	38	−20

Interchanging the coefficients of x and y in F_1
and changing the sign for one of them

	5	6		0	−22	76
b	38		57	0	722	−380
c	25	30		0	−110	+380
d	63	30	57	0	612	0
	76·3	36·3	69·0	0	742	0

WORKING TABLE

Initial values	x	y	z	F_1	F_2	F_3
	0	0	0	+24	+76	+320
Operat.	Δx	Δy	Δz	F_1	F_2	F_3
	2			−24	20	16
				0	96	336
	34		51	0	646	−340
				0	742	−4
	−76·3	−36·3	−69·0	0	742	0
Σ	−40·3	−36·3	−18·0	0	0	−4

Final values: $x = -40·3$; $y = -36·3$; $z = -18·0$.

tial increments. Let the change in a second residual due to the above groups be λ and μ respectively. A combination of these two groups, in the proportion of $-\lambda$ and μ respectively, liquidates the second residual. It may be noted that the value of the original liquidated residual is still unaltered from its zero value during this operation. This is brought out in the solution

of equations as in Table 1. The preparation of the operation table is herein explained. In operation (a) the variable x is adjusted to liquidate the

TABLE 2

Equations:

$$-9x+9y+6z+37 = 0 \quad \equiv F_1$$
$$9x-10y+7z+161 = 0 \quad \equiv F_2$$
$$6x+7y-9z+341 = 0 \quad \equiv F_3$$

OPERATION TABLE

Δx	Δy	Δz	ΔF_1	ΔF_2	ΔF_3
1			-9	9	6
	1		9	-10	7
		1	6	7	-9
1	1	1	6	6	4
1		1	-3	16	-3

WORKING TABLE

Initial values	x	y	z	F_1	F_2	F_3
	0	0	0	37	161	341
Operat.	Δx	Δy	Δz	F_1	F_2	F_3
			38	228	266	-342
				265	427	-1
		43		387	-430	301
				652	-3	300
	72			-648	648	432
				4	645	732
		30		270	-300	210
				274	345	942
			2	12	14	-18
				286	359	924
	20			-180	180	120
				106	539	1044
	1	1	1	6	6	4
				112	545	1048
		1		9	-10	7
				121	535	1055
	6	6		0	-6	78
Σ	99	81	41	121	529	1133
Check	99	81	41	121	529	1133

$n: 0.307$

Multiplying factor: $\dfrac{n}{n-1} = \dfrac{0.307}{0.307-1} = -\dfrac{0.307}{0.693} = -0.443.$

Answers: $-0.443(99, 81, 41) = -43.4, -35.9, -18.2.$

Final values: $x = -43.4; y = -35.9; z = -18.2.$

residual F_1; operation (b), wherein x and z coefficients are interchanged and chosen on the basis as outlined above, keeps this residual unchanged at its

zero value and gives changes in the other two residuals; operation (c), wherein x and y coefficients are interchanged and the group similarly chosen, keeps the liquidated residual at its zero value and gives changes in the other two. Operation (d) shows a combination of (b) and (c) in a definite proportion, depending on the values of second residual, which keeps F_3 in addition to F_1 unchanged. The working table is self-explanatory.

TABLE 3

Equations:

$$-9x+9y+6z+37 = 0 \quad \equiv F_1$$
$$9x-10y+7z+161 = 0 \quad \equiv F_2$$
$$6x+7y-9z+341 = 0 \quad \equiv F_3$$

OPERATION TABLE

Δx	Δy	Δz	ΔF_1	ΔF_2	ΔF_3
1			-9	9	6
	1		9	-10	7
		1	6	7	-9
1	1		0	-1	13
	-2	3	0	41	-41
41	41		0	-41	533
	-26	39	0	533	-533
41	15	39	0	492	0
-18.8	-7.4	-17.8	0	-225	0

WORKING TABLE

Initial values	x	y	z	F_1	F_2	F_3
	0	0	0	37	161	341
Operat.	Δx	Δy	Δz	F_1	F_2	F_3
	4			-36	36	24
				1	197	365
	-28	-28		0	28	-364
				1	225	1
	-18.8	-7.4	-17.8	0	-225	0
Σ	-42.8	-35.4	-17.8	1	0	1

Final values: $x = -42.8$; $y = -35.4$; $z = -17.8$.

Table 2 works out a set of simultaneous linear equations by the method of multiplying factor. Table 3 works out the same problem by the method suggested herein. It may be noted that the difference in the final answers is not great and therefore not worthy of any serious consideration. The answers, it may be seen, agree to significant whole numbers. Table 4 illustrates the algebraic equivalence of the process.

TABLE 4

Equations $a_1x + b_1y + c_1z + d_1 = 0 \equiv F_1$; $a_2x + b_2y + c_2z + d_2 = 0 \equiv F_2$; $a_3x + b_3y + c_3z + d_3 = 0 \equiv F_3$

Algebraic equivalence of the process. To get a fuller demonstration of the method an algebraic equation is solved here. A look at the table makes it clear that the process is rather complicated algebraically and altogether different from the process of successive elimination.

	Δx	Δy	Δz	ΔF_1	ΔF_2	ΔF_3
	1	1	1	a_1	a_2	a_3
				b_1	b_2	b_3
				c_1	c_2	c_3
	b_1	$\dfrac{c_1}{-a_1}$	$\dfrac{1}{-b_1}$	0	$\dfrac{(b_3c_1-c_3b_1)}{(a_3b_1-b_3a_1)(a_2b_1-b_2a_1)}$	$\dfrac{(b_3c_1-c_3b_1)}{(a_3b_1-b_3a_1)(a_2b_1-b_2a_1)}$
	$-(b_2c_1-c_2b_1)b_1$	$(a_2b_1-b_2a_1)c_1 + (b_2c_1-c_2b_1)a_1$	$-(a_2b_1-b_2a_1)$	0		
	$-(b_2c_1-c_2b_1)b_1$	$(a_2b_1-b_2a_1)b_1$	$-(a_2b_1-b_2a_1)$	0	$(b_2c_1-c_2b_1)(a_2b_1-b_2a_1)$	$(a_3b_1-b_3a_1)(a_2b_1-b_2a_1)$
				0	$-(b_2c_1-c_2b_1)(a_3b_1-b_3a_1)$	$-(b_2c_1-c_2b_1)(a_3b_1-b_3a_1)$
				0	0	
Initial values				ΔF_1	ΔF_2	ΔF_3
	x	y	z	d_1	d_2	d_3
(1)	0	0	0	d_1	d_2	d_3
	Δx	Δy	Δz			
	$-\dfrac{d_1}{a_1}$			$-d_1$	$-\dfrac{d_1}{-a_2a_1}$	$-\dfrac{d_1}{-a_3a_1}$
(2)	$\dfrac{c_1}{(b_2c_1-c_2b_1)}\left[-\dfrac{(d_2a_1-a_2d_1)}{a_1}\right]$	$\dfrac{b_1}{(b_2c_1-c_2b_1)}\left[-\dfrac{(d_2a_1-a_2d_1)}{a_1}\right]$	0	0	$\dfrac{d_2a_1-a_2d_1}{a_1}$	$\dfrac{(d_2a_1-a_2d_1)}{a_1(b_2c_1-c_2b_1)}(b_3c_1-c_3b_1)$
					$-\dfrac{d_2a_1-a_3d_1}{a_1}$	$+\dfrac{(d_3a_1-a_3d_1)}{a_1(b_2c_1-c_2b_1)}(b_3c_1-c_3b_1)$
(3)	$-(b_2c_1-c_2b_1)b_1$	$\{(a_2b_1-b_2a_1)c_1 + (b_2c_1-c_2b_1)a_1 - (a_2b_1-b_2a_1)d_1 \} \times \dfrac{\{(a_2b_1-b_2a_1)(d_3a_1-a_3d_1)-(a_2b_1-b_2a_1)b_1\}}{a_1(b_2c_1-c_2b_1)}$	$\dfrac{b_1}{(b_2c_1-c_2b_1)}\left[-\dfrac{(d_2a_1-a_2d_1)}{a_1}\right]$	0	0	$\dfrac{(a_2d_2-a_2d_1)}{a_1} + \dfrac{(d_2a_1-a_2d_1)}{a_1(b_2c_1-c_2b_1)}(b_3c_1-c_3b_1)$
		$\{(a_2b_1-b_2a_1)(b_2c_1-c_2b_1)-(b_2c_1-c_2b_1)(a_2b_1-b_2a_1)\}$				$-\{a_2d_2-a_3d_1\} + \dfrac{d_2a_1-a_2d_1}{a_1(b_2c_1-c_2b_1)}(b_3c_1-c_3b_1)\}$
				0	0	0
Final values	$\Sigma 1, 2, 3$	$\Sigma 1, 2, 3$	Δz	0	0	0

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3. Geometrical interpretation of the process

A simple geometrical interpretation of the process can be easily suggested. Three linear simultaneous equations represent three planes in euclidean space. The object of the relaxation method is an attempt to reach the point of intersection of the planes, always trying to remain as near all the planes as possible and also trying to reduce the distances from them. In this process the groups chosen are equivalent to the displacements parallel to a particular plane as these groups keep one of the residuals unchanged. Similarly, when a group keeps two of the residuals unchanged, it is equivalent to a displacement parallel to the line of intersection of two planes. The whole process, therefore, can be summarized as below:

(i) movement to the surface of any one of the planes;

(ii) movement to the line of intersection of this plane with any other plane and still remaining on the first plane;

(iii) movement to the common point of intersection of all the planes by remaining on the line reached in process (ii).

Previous geometrical interpretation (2) may be mentioned in this connexion.

4. Modifications of the method

This method can be modified by the choice of groups which causes two small displacements of opposite sign instead of one which keeps the residual intact. For example the groups

$$x = 3, \qquad y = 5; \qquad x = 1, \qquad y = -2$$

can be used in the equation

$$100x + 54y + 81z = 300$$

instead of a single group

$$x = 54, \qquad y = -100$$

which has too large numerical values and may, therefore, set in inconvenient fractions in the working table if chosen to liquidate a second residual which is not divisible by 54.

It may also be possible in some equations to see a pair of values of two variables which may liquidate two residuals simultaneously even though these values may be far removed from the solution. In such cases the process begins a step ahead from the normal procedure. An example of this kind can easily be constructed, viz.

$$21x + 30y - 51z - 51 = 0,$$
$$x + y - 2z - 2 = 0,$$
$$x + y - 5 = 0.$$

These will have the first two residuals zero with $x = 1$, $y = 1$.

5. Remarks

The attention of the authors (after the completion of the present work) has been directed to a paper (3), which was similar in spirit to the present work. The authors are not, however, aware of the method suggested herein having been described elsewhere. The method may be extremely suitable for ill-conditioned equations.

6. Acknowledgements

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